

Dynamics of Ising models near zero temperature : Real Space Renormalization Approach

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We consider the stochastic dynamics of Ising ferromagnets (either pure or random) near zero temperature. The master equation satisfying detailed balance can be mapped onto a quantum Hamiltonian which has an exact zero-energy ground state representing the thermal equilibrium. The largest relaxation time t_{eq} governing the convergence towards this Boltzmann equilibrium in finite-size systems is determined by the lowest non-vanishing eigenvalue $E_1 = 1/t_{eq}$ of the quantum Hamiltonian H . We introduce and study a real-space renormalization procedure for the quantum Hamiltonian associated to the single-spin-flip dynamics of Ising ferromagnets near zero temperature. We solve explicitly the renormalization flow for two cases. (i) For the one-dimensional random ferromagnetic chain with free boundary conditions, the largest relaxation time t_{eq} can be expressed in terms of the set of random couplings for various choices of the dynamical transition rates. The validity of these RG results in $d = 1$ is checked by comparison with another approach. (ii) For the pure Ising model on a Cayley tree of branching ratio K , we compute the exponential growth of $t_{eq}(N)$ with the number N of generations.

I. INTRODUCTION

The stochastic dynamics of classical Ising ferromagnets has been much studied for fifty years [1, 2]. In particular, many works have been devoted to the domain growth dynamics at low temperature $T < T_c$ (or at zero temperature $T = 0$ in $d = 1$ where the critical temperature vanishes $T_c = 0$) when the initial condition is random (see the review on phase ordering dynamics [3]).

In the present paper, we do not consider the dynamics starting from a random initial condition, but we focus instead on the largest relaxation time t_{eq} needed for a finite systems to converge towards thermal equilibrium. This largest relaxation time t_{eq} is defined as the inverse of the smallest non-vanishing eigenvalue E_1 of the time-evolution operator. Near zero-temperature, more precisely when the temperature is much smaller than any ferromagnetic coupling J_{ij}

$$0 < T \ll J_{ij} \quad (1)$$

the thermal equilibrium is dominated by the two ferromagnetic ground states where all spins take the same value, and the largest relaxation time t_{eq} corresponds to the time needed to go from one ground state (where all spins take the value $+1$) to the opposite ground state (where all spins take the value -1). We should stress that we consider that the temperature is arbitrarily small, but does not vanish, so that the transition between the two ground states is possible and the final state of the dynamics is unique (For studies on the zero-temperature dynamics, where the spin-flips corresponding to an energy-increase become impossible, we refer to the recent works [4] and to references therein).

Of course near zero temperature, the equilibration time t_{eq} becomes extremely large, and numerical simulations of the microscopic dynamics become inefficient. Here we introduce and study a real-space renormalization procedure valid near zero temperature for the dynamics of pure or random ferromagnets. The paper is organized as follows. In section II, we recall the standard mapping between the master equation describing the stochastic dynamics of classical systems at temperature T and a special type of quantum Hamiltonians that have an exact zero-energy eigenstate. In section III, we describe the various choices of dynamical transition rates for single-spin-flip dynamics of the classical Ising model, and the corresponding quantum Hamiltonians. In section IV, we introduce the real-space renormalization procedure for these general quantum Hamiltonians. In section V, we show how a closed RG procedure can be defined and exactly solved for the random ferromagnetic chain. In section VI, we solve the RG flow for the pure Ising model on a Cayley tree. Section VII summarizes our conclusions. In Appendix A, we describe another approach that allows to check the validity of the RG results of section V. Finally in Appendix B, we discuss the contributions that can depend on the choice of transition rates satisfying detailed balance.

II. RELAXATION OF CLASSICAL MODELS TOWARDS THERMAL EQUILIBRIUM

A. Master Equation satisfying detailed balance

To define the stochastic dynamics of a classical system, it is convenient to consider the master equation

$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} P_t(\mathcal{C}') W(\mathcal{C}' \rightarrow \mathcal{C}) - P_t(\mathcal{C}) W_{out}(\mathcal{C}) \quad (2)$$

that describes the time evolution of the probability $P_t(\mathcal{C})$ to be in configuration \mathcal{C} at time t . The notation $W(\mathcal{C}' \rightarrow \mathcal{C})$ represents the transition rate per unit time from configuration \mathcal{C}' to \mathcal{C} , and

$$W_{out}(\mathcal{C}) \equiv \sum_{\mathcal{C}'} W(\mathcal{C} \rightarrow \mathcal{C}') \quad (3)$$

represents the total exit rate out of configuration \mathcal{C} .

For a classical system where each configuration \mathcal{C} has some energy $U(\mathcal{C})$, the convergence towards Boltzmann equilibrium at temperature $T = \frac{1}{\beta}$ in any finite system

$$P_{eq}(\mathcal{C}) = \frac{e^{-\beta U(\mathcal{C})}}{Z} \quad (4)$$

where Z is the partition function

$$Z = \sum_{\mathcal{C}} e^{-\beta U(\mathcal{C})} \quad (5)$$

can be ensured by imposing the detailed balance property

$$e^{-\beta U(\mathcal{C})} W(\mathcal{C} \rightarrow \mathcal{C}') = e^{-\beta U(\mathcal{C}')} W(\mathcal{C}' \rightarrow \mathcal{C}) \quad (6)$$

B. Mapping onto a Schrödinger equation in configuration space

As is well known (see for instance the textbooks [5–7]), the non-symmetric operator describing the stochastic dynamics of a classical model at temperature T can be transformed into a symmetric quantum Hamiltonian problem. In the field of disordered systems, this mapping has been much used for one-dimensional continuous models [8–11], and more recently for many-body spin systems like the Sherrington-Kirkpatrick model ([12] and Appendix B of [13]). In the field of pure spin models, this mapping has been used for more than fifty years [2, 14–17].

In the present context, this standard mapping consists in the change of variable

$$P_t(\mathcal{C}) \equiv e^{-\frac{\beta}{2} U(\mathcal{C})} \psi_t(\mathcal{C}) = e^{-\frac{\beta}{2} U(\mathcal{C})} \langle \mathcal{C} | \psi_t \rangle \quad (7)$$

Then the master equation of Eq. 2 becomes the imaginary-time Schrödinger equation for the ket $|\psi_t\rangle$

$$\frac{d}{dt} |\psi_t\rangle = -H |\psi_t\rangle \quad (8)$$

where the quantum Hamiltonian

$$\mathcal{H} = \sum_{\mathcal{C}} \epsilon(\mathcal{C}) |\mathcal{C}\rangle \langle \mathcal{C}| + \sum_{\mathcal{C}, \mathcal{C}'} V(\mathcal{C}, \mathcal{C}') |\mathcal{C}'\rangle \langle \mathcal{C}| \quad (9)$$

contains the on-site energies

$$\epsilon(\mathcal{C}) = W_{out}(\mathcal{C}) \equiv \sum_{\mathcal{C}'} W(\mathcal{C} \rightarrow \mathcal{C}') \quad (10)$$

and the hoppings (using Eq 6)

$$\begin{aligned} V(\mathcal{C}, \mathcal{C}') &= -e^{-\frac{\beta}{2}[U(\mathcal{C}') - U(\mathcal{C})]} W(\mathcal{C}' \rightarrow \mathcal{C}) = -e^{-\frac{\beta}{2}[U(\mathcal{C}) - U(\mathcal{C}')] } W(\mathcal{C} \rightarrow \mathcal{C}') \\ &= -\sqrt{W(\mathcal{C} \rightarrow \mathcal{C}') W(\mathcal{C}' \rightarrow \mathcal{C})} \end{aligned} \quad (11)$$

C. Properties of the quantum Hamiltonian \mathcal{H}

Let us note E_n the eigenvalues of the quantum Hamiltonian \mathcal{H} and $|\psi_n\rangle$ the associated normalized eigenvectors

$$\mathcal{H}|\psi_n\rangle = E_n|\psi_n\rangle \quad (12)$$

$$\sum_{\mathcal{C}} |\psi_n(\mathcal{C})|^2 = 1 \quad (13)$$

The evolution operator $e^{-t\mathcal{H}}$ can be expanded on the eigenstates

$$e^{-t\mathcal{H}} = \sum_n e^{-E_n t} |\psi_n\rangle \langle \psi_n| \quad (14)$$

The conditional probability $P_t(\mathcal{C}|\mathcal{C}_0)$ to be in configuration \mathcal{C} at t if one starts from the configuration \mathcal{C}_0 at time $t = 0$ can be written as

$$P_t(\mathcal{C}|\mathcal{C}_0) = e^{-\frac{\beta}{2}[U(\mathcal{C})-U(\mathcal{C}_0)]} \langle \mathcal{C} | e^{-t\mathcal{H}} | \mathcal{C}_0 \rangle = e^{-\frac{\beta}{2}[U(\mathcal{C})-U(\mathcal{C}_0)]} \sum_n e^{-E_n t} \psi_n(\mathcal{C}) \psi_n^*(\mathcal{C}_0) \quad (15)$$

The quantum Hamiltonian \mathcal{H} has special properties that come from its relation to the dynamical master equation :

(i) the ground state energy is $E_0 = 0$, and the corresponding eigenvector is given by

$$|\psi_0\rangle = \sum_{\mathcal{C}} \frac{e^{-\frac{\beta}{2}U(\mathcal{C})}}{\sqrt{Z}} |\mathcal{C}\rangle \quad (16)$$

The normalization $1/\sqrt{Z}$ comes from the quantum normalization of Eq. 13.

This property ensures the convergence towards the Boltzmann equilibrium in Eq. 7 for any initial condition \mathcal{C}_0

$$P_t(\mathcal{C}|\mathcal{C}_0) \underset{t \rightarrow +\infty}{\simeq} e^{-\frac{\beta}{2}[U(\mathcal{C})-U(\mathcal{C}_0)]} \psi_0(\mathcal{C}) \psi_0^*(\mathcal{C}_0) = \frac{e^{-\beta U(\mathcal{C})}}{Z} = P_{eq}(\mathcal{C}) \quad (17)$$

(ii) the other energies $E_n > 0$ determine the relaxation towards equilibrium. In particular, the lowest non-vanishing energy E_1 determines the largest relaxation time ($1/E_1$) of the system

$$P_t(\mathcal{C}|\mathcal{C}_0) - P_{eq}(\mathcal{C}) \underset{t \rightarrow +\infty}{\simeq} e^{-E_1 t} e^{-\frac{\beta}{2}[U(\mathcal{C})-U(\mathcal{C}_0)]} \psi_1(\mathcal{C}) \psi_1^*(\mathcal{C}_0) \quad (18)$$

Since this largest relaxation time represents the 'equilibrium time', i.e. the characteristic time needed to converge towards equilibrium, we will use the following notation

$$t_{eq} \equiv \frac{1}{E_1} \quad (19)$$

In summary, the relaxation time t_{eq} can be computed without simulating the dynamics by any method able to compute the first excited energy E_1 of the quantum Hamiltonian \mathcal{H} (where the ground state is given by Eq. 16 and has for eigenvalue $E_0 = 0$). For instance in [12], the 'conjugate gradient' method has been used to study numerically the statistics of the largest relaxation time in various disordered models. Let us now describe more precisely how this general framework applies to single-spin-flip dynamics of Ising models.

III. QUANTUM HAMILTONIAN ASSOCIATED TO SINGLE-SPIN-FLIP DYNAMICS

A. Single-spin-flip dynamics

We consider a system of classical spins $S_i = \pm 1$ where each configuration $\mathcal{C} = S_1, S_2, \dots$ has for energy

$$U(\mathcal{C}) = - \sum_{i < j} J_{ij} S_i S_j \quad (20)$$

The couplings J_{ij} may be random.

Within a single-spin flip dynamics, the configuration $|\mathcal{C}\rangle = |S_1\rangle |S_2\rangle \dots |S_N\rangle$ containing N spins is connected via the transition rates $W(\mathcal{C} \rightarrow \mathcal{C}')$ to the N configurations obtained by the flip of a single spin $S_k \rightarrow -S_k$ denoted by

$$|\mathcal{C}_k\rangle = \sigma_k^x |\mathcal{C}\rangle \quad (21)$$

in terms of the Pauli matrix σ_x . The energy difference between the two configurations reads

$$U(\mathcal{C}_k) - U(\mathcal{C}) = 2S_k \sum_i J_{ki} S_i \quad (22)$$

The quantum Hamiltonian of Eqs 9 10 11 can be thus rewritten as

$$\begin{aligned} \mathcal{H} &= \sum_{\mathcal{C}} \epsilon(\mathcal{C}) |\mathcal{C}\rangle \langle \mathcal{C}| + \sum_{\mathcal{C}} \sum_{k=1}^N V(\mathcal{C}, \sigma_k^x \mathcal{C}) \sigma_k^x |\mathcal{C}\rangle \langle \mathcal{C}| \\ &= \sum_{\mathcal{C}} \sum_{k=1}^N W(\mathcal{C} \rightarrow \sigma_k^x \mathcal{C}) e^{\frac{\beta}{2}[U(\mathcal{C}_k) - U(\mathcal{C})]} \left[e^{-\frac{\beta}{2}[U(\mathcal{C}_k) - U(\mathcal{C})]} - \sigma_k^x \right] |\mathcal{C}\rangle \langle \mathcal{C}| \end{aligned} \quad (23)$$

B. Simplest choice of the transition rates

It is clear from Eq. 23 that the simplest quantum Hamiltonian corresponds to the following choice of the dynamical transition rate

$$W(\mathcal{C} = \{S_i\} \rightarrow \mathcal{C}_k = \sigma_k^x \mathcal{C}) = e^{-\frac{\beta}{2}[U(\mathcal{C}_k) - U(\mathcal{C})]} = e^{-\beta S_k (\sum_{i \neq k} J_{ik} S_i)} \quad (24)$$

The quantum Hamiltonian of Eq. 23 then reads in terms of Pauli matrices

$$\begin{aligned} \mathcal{H}^{simple} &= \sum_{\mathcal{C}} \sum_{k=1}^N \left[e^{-\frac{\beta}{2}[U(\mathcal{C}_k) - U(\mathcal{C})]} - \sigma_k^x \right] |\mathcal{C}\rangle \langle \mathcal{C}| \\ &= \sum_{\mathcal{C}} \sum_{k=1}^N \left[e^{-\beta S_k (\sum_i J_{ki} S_i)} - \sigma_k^x \right] |\mathcal{C}\rangle \langle \mathcal{C}| \\ &= \sum_{k=1}^N \left[e^{-\beta \sigma_k^z (\sum_{i \neq k} J_{ik} \sigma_i^z)} - \sigma_k^x \right] \end{aligned} \quad (25)$$

where we have used the identity $1 = \sum_{\mathcal{C}} |\mathcal{C}\rangle \langle \mathcal{C}|$. The quantum Hamiltonian of Eq. 25 has been mentioned as the simplest for the one-dimensional ferromagnetic chain in Eq (4) of Ref [15].

C. Glauber choice

The Glauber choice for the transition rates [2]

$$W(\mathcal{C} = \{S_i\} \rightarrow \mathcal{C}_k = \sigma_k^x \mathcal{C}) = \frac{e^{-\frac{\beta}{2}[U(\mathcal{C}_k) - U(\mathcal{C})]}}{2 \cosh\left(\frac{\beta}{2}[U(\mathcal{C}_k) - U(\mathcal{C})]\right)} = \frac{e^{-\beta S_k (\sum_{i \neq k} J_{ik} S_i)}}{2 \cosh\left(\beta \left[\sum_{i \neq k} J_{ik} S_i\right]\right)} \quad (26)$$

corresponds to the more complicated quantum Hamiltonian

$$H^{Glauber} = \sum_{k=1}^N \frac{1}{2 \cosh\left[\beta \left(\sum_{i \neq k} J_{ik} \sigma_i^z\right)\right]} \left(e^{-\beta \sigma_k^z (\sum_{i \neq k} J_{ik} \sigma_i^z)} - \sigma_k^x \right) \quad (27)$$

where we have used the fact that σ_k^z has for eigenvalues (± 1) and that \cosh is an even function.

This quantum Hamiltonian of Eq. 27 has been already used for the Sherrington-Kirkpatrick spin-glass model and for the finite dimensional ferromagnetic Ising model in Ref. [13] (see Appendix B and Appendix C respectively).

For the one-dimensional pure ferromagnetic chain, where each spin S_k has only two neighbors, the local field $B_k \equiv \sum_i J_{ik} S_i = J(S_{k-1} + S_{k+1})$ can take only the three values $h_k = -2J, 0, 2J$, so that one may replace the exponential factors using projection operators to recover the forms given in Refs [15–17]. This type of ‘first quantized’ quantum spin Hamiltonian can be transformed further into ‘second quantized’ Hamiltonian involving annihilation/creation operators or Fermi operators using Jordan-Wigner transformation (see the review [18]) : this method has been followed in particular by Ref. [14] for the Glauber dynamics of the pure Ising chain. However in the present paper, we will work directly on the ‘first quantized’ form of the quantum spin Hamiltonian of Eq 25 or Eq 27, with the aim to define a real-space renormalization approach, in analogy with the Strong Disorder RG rules introduced for the random transverse field Ising model on its first-quantized form (see the discussion in section IV A below).

D. Most general choice

To better understand the structure of the renormalized Hamiltonian that will be generated by the real space RG procedure introduced in the following section, it is useful to consider the most general choice satisfying the detailed balance equation of Eq. 6

$$W(\mathcal{C} = \{S_i\} \rightarrow \mathcal{C}_k = \sigma_k^x \mathcal{C}) = G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) e^{-\beta S_k [\sum_{i \neq k} J_{ik} S_i]} \quad (28)$$

where $G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N)$ is an arbitrary positive function of the $(N-1)$ spins $i \neq k$ that may depend on the index k . The positivity requirement

$$G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) > 0 \quad (29)$$

ensures that all elementary single-flip are possible with a non-vanishing rate, so that the ground state of Eq. 16 is unique, and the dynamics converges towards thermal equilibrium. Since the reversed transition of Eq. 28 has the following rate

$$W(\mathcal{C}_k = \sigma_k^x \mathcal{C} \rightarrow \mathcal{C} = \{S_i\}) = G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) e^{+\beta S_k [\sum_{i \neq k} J_{ik} S_i]} \quad (30)$$

the amplitude $G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N)$ represents the ‘symmetric part’ of the two opposite transitions involving the flip of the spin k (when all other spins remain fixed)

$$G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) = \sqrt{W(\mathcal{C} \rightarrow \sigma_k^x \mathcal{C}) W(\sigma_k^x \mathcal{C} \rightarrow \mathcal{C})} \quad (31)$$

The corresponding quantum Hamiltonian of Eq 23 reads

$$\mathcal{H}^{general} = \sum_{k=1}^N G_k(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) \left(e^{-\beta \sigma_k^z (\sum_{i \neq k} J_{ik} \sigma_i^z)} - \sigma_k^x \right) \quad (32)$$

In finite dimensions with only nearest-neighbors interactions, it seems natural to consider local rates where the amplitude only involves the spins i that are neighbors of k (i.e. such that $J_{ki} \neq 0$)

$$G_k(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) \rightarrow G_k^{local}(\{\sigma_i^z\}_{J_{ik} \neq 0}) \quad (33)$$

To respect the symmetries of the classical energy of Eq. 20, a further requirement is usually that the amplitude should only be an even function $G(x) = G(-x)$ of the local field

$$G_k^{local}(\{\sigma_i^z\}_{J_{ik} \neq 0}) \rightarrow G\left(\sum_i J_{ki} \sigma_i^z\right) \quad (34)$$

In the next section, we introduce a renormalization procedure for the amplitude G_k and we will see that even if one starts with a function of the local field (Eq. 34), this form is not stable via renormalization and will generally lead to a local function of the neighboring spins (Eq. 33).

IV. RENORMALIZATION RULES NEAR ZERO TEMPERATURE

In this section, we derive the appropriate renormalization rules for the quantum Hamiltonian introduced in the previous section in relation with the single-spin flip dynamics of classical ferromagnets,

A. Differences with the disordered quantum Ising model

Let us first mention that in the high-temperature limit $\beta J_{i,j} = \frac{J_{i,j}}{T} \ll 1$, the quantum Hamiltonian of Eq. 25 or 27 reduces to the standard transverse-field Ising model

$$H^{simple, Glauber} \underset{\beta J_{i,j} \ll 1}{\simeq} N + \sum_{k=1}^N \left(-\beta \sigma_k^z \left(\sum_{i \neq k} J_{ik} \sigma_i^z \right) - \sigma_k^x \right) \quad (35)$$

When the coupling J_{ij} are random, the low-energy physics is then well described by the Strong Disorder RG procedure valid both in one dimension [19] and in higher dimensions $d > 1$ [20–22] (see [23] for a review). However here we are interested into the opposite limit of very low temperature where $\beta J_{i,j} = \frac{J_{i,j}}{T} \gg 1$ (Eq 1), where one cannot linearize the exponentials in the quantum Hamiltonians of Eq. 25 and 27. In the following, we thus derive appropriate RG rules in this opposite regime.

B. Analysis of an elementary operator

The general quantum Hamiltonian of Eq 32 can be considered as a sum

$$\mathcal{H}^{general} = \sum_{k=1}^N h_k \quad (36)$$

of elementary operators

$$h_k \equiv G_k(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) \left(e^{-\beta \sigma_k^z (\sum_i J_{ki} \sigma_i^z)} - \sigma_k^x \right) \quad (37)$$

The corresponding matrix elements in the σ^z basis

$$\langle S'_1, \dots, S'_N | h_k | S_1, \dots, S_N \rangle = \left(\prod_{j \neq k} \delta_{S'_j, S_j} \right) G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) \langle S'_k | e^{-\beta \sigma_k^z (\sum_{i \neq k} J_{ki} S_i)} - \sigma_k^x | S_k \rangle \quad (38)$$

are diagonal for all spins $j \neq k$.

1. Effective problem for a single spin

For each fixed value of the local field $B_k \equiv \sum_i J_{ik} S_i$, we may diagonalize the effective problem for the single-spin k

$$h_k^{eff}(B_k) \equiv e^{-\beta \sigma_k^z B_k} - \sigma_k^x \quad (39)$$

The two normalized eigenstates are respectively

$$|v_k(B_k)\rangle \equiv \frac{1}{\sqrt{2 \cosh(\beta B_k)}} \left[e^{\frac{\beta}{2} B_k} |S_k = +1\rangle + e^{-\frac{\beta}{2} B_k} |S_k = -1\rangle \right] \quad (40)$$

and

$$|w_k(B_k)\rangle \equiv \frac{1}{\sqrt{2 \cosh(\beta B_k)}} \left[e^{-\frac{\beta}{2} B_k} |S_k = +1\rangle - e^{\frac{\beta}{2} B_k} |S_k = -1\rangle \right] \quad (41)$$

with eigenvalues

$$\begin{aligned} h_k^{eff}(B_k) |v_k(B_k)\rangle &= 0 \\ h_k^{eff}(B_k) |w_k(B_k)\rangle &= 2 \cosh(\beta B_k) |w_k(B_k)\rangle \end{aligned} \quad (42)$$

so that the single-spin hamiltonian of Eq. 39 can be rewritten as the projector

$$\begin{aligned}
h_k^{eff}(B_k) &= 2 \cosh(\beta B_k) |w_k(B_k) \rangle \langle w_k(B_k)| \\
&= \left[e^{-\frac{\beta}{2} B_k} |S_k = +1 \rangle - e^{\frac{\beta}{2} B_k} |S_k = -1 \rangle \right] \left[e^{-\frac{\beta}{2} B_k} \langle S_k = +1| - e^{\frac{\beta}{2} B_k} \langle S_k = -1| \right] \\
&= \sigma_k^z e^{-\frac{\beta}{2} \sigma_k^z B_k} [|S_k = +1 \rangle + |S_k = -1 \rangle] [\langle S_k = +1| + \langle S_k = -1|] \sigma_k^z e^{-\frac{\beta}{2} \sigma_k^z B_k}
\end{aligned} \tag{43}$$

In terms of operators, the elementary operator of Eq. 37 may be thus rewritten as

$$\begin{aligned}
h_k &= G_k(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) \\
&\sigma_k^z e^{-\frac{\beta}{2} \sigma_k^z \sum_i J_{ki} \sigma_i^z} [|S_k = +1 \rangle + |S_k = -1 \rangle] [\langle S_k = +1| + \langle S_k = -1|] \sigma_k^z e^{-\frac{\beta}{2} \sigma_k^z \sum_i J_{ki} \sigma_i^z}
\end{aligned} \tag{44}$$

2. Properties of elementary operators

Eq 44 is convenient to see explicitly the positivity property for any ket $|\psi \rangle$

$$\begin{aligned}
\langle \psi | h_k | \psi \rangle &= \sum_{S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N} G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) \\
&|e^{-\frac{\beta}{2} \sum_i J_{ki} S_i} \psi(S_1, \dots, S_{k-1}, S_k = +1, S_{k+1}, \dots, S_N) - e^{\frac{\beta}{2} \sum_i J_{ki} S_i} \psi(S_1, \dots, S_{k-1}, S_k = -1, S_{k+1}, \dots, S_N)|^2 \\
&\geq 0
\end{aligned} \tag{45}$$

as a consequence of the positivity of the G_k (Eq 29). Moreover, it is clear that the exactly known ground state of zero energy of Eq. 16

$$\begin{aligned}
|\psi_0 \rangle &= \sum_{S_1, \dots, S_N} \langle S_1, \dots, S_N | \psi_0 \rangle |S_1, \dots, S_N \rangle = \frac{1}{\sqrt{Z_N(\beta)}} \sum_{S_1, \dots, S_N} e^{\frac{\beta}{2} \sum_{1 \leq i < j \leq N} J_{ij} S_i S_j} |S_1, \dots, S_N \rangle \\
Z_N(\beta) &= \sum_{S_1, \dots, S_N} e^{\beta \sum_{1 \leq i < j \leq N} J_{ij} S_i S_j}
\end{aligned} \tag{46}$$

satisfies

$$\frac{\psi_0(S_1, \dots, S_{k-1}, S_k = +1, S_{k+1}, \dots, S_N)}{\psi_0(S_1, \dots, S_{k-1}, S_k = -1, S_{k+1}, \dots, S_N)} = e^{\beta \sum_i J_{ki} S_i} \tag{47}$$

and is thus annihilated by all elementary operators h_k for $k = 1, \dots, N$

$$h_k |\psi_0 \rangle = 0 \tag{48}$$

as it should.

C. Renormalization of the sum of two neighboring elementary operators

1. Sum of two neighboring elementary operators

The sum of two neighboring local operators of Eq. 37 of index k and l with $J_{kl} \neq 0$ reads

$$\begin{aligned}
h_k + h_l &= G_k(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) \left(e^{-\beta \sigma_k^z (J_{kl} \sigma_l^z + \sum_{i \neq l} J_{ki} \sigma_i^z)} - \sigma_k^x \right) \\
&+ G_l(\sigma_1^z, \dots, \sigma_{l-1}^z, \sigma_{l+1}^z, \dots, \sigma_N^z) \left(e^{-\beta \sigma_l^z (J_{kl} \sigma_k^z + \sum_{j \neq k} J_{lj} \sigma_j^z)} - \sigma_l^x \right)
\end{aligned} \tag{49}$$

The corresponding matrix elements in the σ^z basis

$$\begin{aligned}
\langle S'_1, \dots, S'_N | h_k + h_l | S_1, \dots, S_N \rangle &= \left(\prod_{j \neq (k, l)} \delta_{S'_j, S_j} \right) \\
&[G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_N) \langle S'_k | e^{-\beta \sigma_k^z (J_{kl} S_l + \sum_{i \neq l} J_{ki} S_i)} - \sigma_k^x | S_k \rangle \\
&+ G_l(S_1, \dots, S_{l-1}, S_{l+1}, \dots, S_N) \langle S'_l | e^{-\beta \sigma_l^z (J_{kl} S_k + \sum_{i \neq k} J_{li} S_i)} - \sigma_l^x | S_l \rangle]
\end{aligned} \tag{50}$$

are diagonal for all spins $j \neq (k, l)$.

2. Effective two-spin problem

For each fixed value of all the other external spins $S_{j \neq (k,l)}$, using the notations

$$\begin{aligned} B_k &\equiv \sum_{i \neq l} J_{ki} S_i \\ B_l &\equiv \sum_{j \neq k} J_{lj} S_j \end{aligned} \quad (51)$$

and

$$\begin{aligned} g_k^{S_l} &\equiv G_k(S_1, \dots, S_{k-1}, S_{k+1}, \dots, S_l, S_N) \\ g_l^{S_k} &\equiv G_l(S_1, \dots, S_k, \dots, S_{l-1}, S_{l+1}, \dots, S_N) \end{aligned} \quad (52)$$

we have to diagonalize the effective problem for the two spins (k, l)

$$h_{k,l}^{eff} \equiv g_k^{\sigma_k^z} \left(e^{-\beta \sigma_k^z (J_{kl} \sigma_l^z + B_k)} - \sigma_k^x \right) + g_l^{\sigma_l^z} \left(e^{-\beta \sigma_l^z (J_{kl} \sigma_k^z + B_l)} - \sigma_l^x \right) \quad (53)$$

The four-dimensional vector

$$|u_\lambda\rangle = \sum_{S_k=\pm, S_l=\pm} c_{\lambda}^{S_k, S_l} |S_k, S_l\rangle \quad (54)$$

is an eigenvector of the operator $h_{k,l}^{eff}$ of Eq. 53 with eigenvalue λ if

$$\begin{aligned} 0 &= [e^{-\beta J_{kl}} (g_k^+ e^{-\beta B_k} + g_l^+ e^{-\beta B_l}) - \lambda] c_{\lambda}^{++} - g_l^+ c_{\lambda}^{+-} - g_k^+ c_{\lambda}^{-+} \\ 0 &= [e^{-\beta J_{kl}} (g_k^- e^{\beta B_k} + g_l^- e^{\beta B_l}) - \lambda] c_{\lambda}^{--} - g_k^- c_{\lambda}^{+-} - g_l^- c_{\lambda}^{-+} \\ 0 &= [e^{\beta J_{kl}} (g_k^- e^{-\beta B_k} + g_l^+ e^{\beta B_l}) - \lambda] c_{\lambda}^{+-} - g_l^+ c_{\lambda}^{++} - g_k^- c_{\lambda}^{--} \\ 0 &= [e^{\beta J_{kl}} (g_k^+ e^{\beta B_k} + g_l^- e^{-\beta B_l}) - \lambda] c_{\lambda}^{-+} - g_k^+ c_{\lambda}^{++} - g_l^- c_{\lambda}^{--} \end{aligned} \quad (55)$$

3. Finding the lowest non-zero eigenvalue λ_1

We already know the exact ground state $|u_{\lambda=0}\rangle$ of eigenvalue $\lambda = 0$, with components (not normalized here)

$$c_{\lambda=0}^{S_k, S_l} = e^{\frac{\beta}{2} J_{kl} S_k S_l + \frac{\beta}{2} B_k S_k + \frac{\beta}{2} B_l S_l} \quad (56)$$

To find the exact three other eigenvalues, one has to solve the remaining cubic equation. Here, since we are interested in the other small eigenvalue $\lambda_1 \ll e^{\beta J_{kl}}$, we will neglect λ_1 in the two last equations of Eq. 55 to obtain

$$\begin{aligned} c_{\lambda}^{+-} &= e^{-\beta J_{kl}} \frac{g_l^+ c_{\lambda}^{++} + g_k^- c_{\lambda}^{--}}{g_k^- e^{-\beta B_k} + g_l^+ e^{\beta B_l}} \\ c_{\lambda}^{-+} &= e^{-\beta J_{kl}} \frac{g_k^+ c_{\lambda}^{++} + g_l^- c_{\lambda}^{--}}{g_k^+ e^{\beta B_k} + g_l^- e^{-\beta B_l}} \end{aligned} \quad (57)$$

that we may replace in the two first equations of Eq. 55

$$\begin{aligned} 0 &= [g_k^+ e^{-\beta B_k} + g_l^+ e^{-\beta B_l} - \lambda_1 e^{\beta J_{kl}}] c_{\lambda}^{++} - g_l^+ \frac{g_l^+ c_{\lambda}^{++} + g_k^- c_{\lambda}^{--}}{g_k^- e^{-\beta B_k} + g_l^+ e^{\beta B_l}} - g_k^+ \frac{g_k^+ c_{\lambda}^{++} + g_l^- c_{\lambda}^{--}}{g_k^+ e^{\beta B_k} + g_l^- e^{-\beta B_l}} \\ 0 &= [g_k^- e^{\beta B_k} + g_l^- e^{\beta B_l} - \lambda_1 e^{\beta J_{kl}}] c_{\lambda}^{--} - g_k^- \frac{g_l^+ c_{\lambda}^{++} + g_k^- c_{\lambda}^{--}}{g_k^- e^{-\beta B_k} + g_l^+ e^{\beta B_l}} - g_l^- \frac{g_k^+ c_{\lambda}^{++} + g_l^- c_{\lambda}^{--}}{g_k^+ e^{\beta B_k} + g_l^- e^{-\beta B_l}} \end{aligned} \quad (58)$$

For $\lambda_1 = 0$, we recover the exact solution of Eq. 56 as it should. The other eigenvalue reads

$$\lambda_1 = e^{-\beta J_{kl}} 2 \cosh[\beta(B_k + B_l)] \frac{g_k^+ g_k^- (g_l^+ e^{\beta B_k} + g_l^- e^{-\beta B_k}) + g_l^+ g_l^- (g_k^+ e^{\beta B_l} + g_k^- e^{-\beta B_l})}{g_k^+ g_k^- + g_l^+ g_l^- + g_k^+ g_l^+ e^{\beta(B_k+B_l)} + g_k^- g_l^- e^{-\beta(B_k+B_l)}} \quad (59)$$

with the corresponding components of the eigenvector $|u_{\lambda_1}\rangle$ (not normalized here)

$$\begin{aligned} c_{\lambda_1}^{++} &= e^{\frac{\beta}{2}J_{kl} - \frac{\beta}{2}B_k - \frac{\beta}{2}B_l} \\ c_{\lambda_1}^{--} &= -e^{\frac{\beta}{2}J_{kl} + \frac{\beta}{2}B_k + \frac{\beta}{2}B_l} \end{aligned} \quad (60)$$

The two other components $c_{\lambda_1}^{+-}$ and $c_{\lambda_1}^{-+}$ are given by Eq 57.

4. Projection onto the two ferromagnetic states

The projection of the operator of Eq. 53 onto its two lowest states of eigenvalues $\lambda_0 = 0$ et λ_1 reads

$$h_{k,l}^{eff} \simeq \frac{\lambda_1}{\langle u_{\lambda_1} | u_{\lambda_1} \rangle} |u_{\lambda_1}\rangle \langle u_{\lambda_1}| \quad (61)$$

At the level of approximation we are working, we wish to keep only the two ferromagnetic states $++$ and $--$ (the two other states $+-$ and $-+$ have been taken into account in Eq. 57 to produce renormalized rates between $++$ and $--$ in Eq 58). So we keep only the two following leading components in the eigenvector

$$|u_{\lambda_1}\rangle \simeq e^{\frac{\beta}{2}J_{kl}} \left(e^{-\frac{\beta}{2}(B_k+B_l)} |++\rangle - e^{\frac{\beta}{2}(B_k+B_l)} |--\rangle \right) \quad (62)$$

with the corresponding normalization

$$\langle u_{\lambda_1} | u_{\lambda_1} \rangle \simeq e^{\beta J_{kl}} 2 \cosh[\beta(B_k + B_l)] \quad (63)$$

Eq 61 becomes

$$\begin{aligned} h_{k,l}^{eff} &\simeq \frac{\lambda_1}{2 \cosh[\beta(B_k + B_l)]} \\ &\left(e^{-\frac{\beta}{2}(B_k+B_l)} |++\rangle - e^{\frac{\beta}{2}(B_k+B_l)} |--\rangle \right) \left(e^{-\frac{\beta}{2}(B_k+B_l)} \langle ++| - e^{\frac{\beta}{2}(B_k+B_l)} \langle --| \right) \end{aligned} \quad (64)$$

So the two spins S_k and S_l now form a single renormalized ferromagnetic cluster, that we may represent by a single spin with the two states

$$\begin{aligned} |+\rangle_R &= |++\rangle \\ |-\rangle_R &= |--\rangle \end{aligned} \quad (65)$$

with the renormalized external field (see Eq 51)

$$B_R = B_k + B_l = \sum_{i \neq (k,l)} (J_{ki} + J_{li}) S_i \quad (66)$$

that corresponds to the natural renormalization of ferromagnetic coupling between the new cluster (kl) and i

$$J_{(kl)i}^R = J_{ki} + J_{li} \quad (67)$$

In terms of this renormalized cluster, Eq. 64 reads

$$h_{k,l}^{eff} \simeq G_R \left(e^{-\frac{\beta}{2}B_R} |+\rangle_R - e^{\frac{\beta}{2}B_R} |-\rangle_R \right) \left(e^{-\frac{\beta}{2}B_R} \langle +|_R - e^{\frac{\beta}{2}B_R} \langle -|_R \right) \quad (68)$$

It has the same form as an elementary operator of Eq. 44 with the renormalized amplitude (using Eq. 59)

$$\begin{aligned} G_R &= \frac{\lambda_1}{2 \cosh[\beta(B_k + B_l)]} \\ &= e^{-\beta J_{kl}} \frac{g_k^+ g_k^- (g_l^+ e^{\beta B_k} + g_l^- e^{-\beta B_k}) + g_l^+ g_l^- (g_k^+ e^{\beta B_l} + g_k^- e^{-\beta B_l})}{g_k^+ g_k^- + g_l^+ g_l^- + g_k^+ g_l^+ e^{\beta B_R} + g_k^- g_l^- e^{-\beta B_R}} \end{aligned} \quad (69)$$

in terms of the variables defined in Eq. 52 for a given value of the external spins $S_{i \neq (k,l)}$.

5. Renormalization rules for operators

At the level of operators in the whole Hilbert space, the renormalized Hamiltonian of Eq. 64 reads (using the same notations as in Eq. 37)

$$h_{(k,l)}^R \equiv G_{kl}^R(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_{l-1}^z, \sigma_{l+1}^z, \dots, \sigma_N^z) \left(e^{-\beta \sigma_R^z B_R} - \sigma_R^x \right) \quad (70)$$

where the amplitude of Eq. 69 is the renormalized operator

$$G_{kl}^R(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_{l-1}^z, \sigma_{l+1}^z, \dots, \sigma_N^z) = e^{-\beta J_{kl}} \frac{g_k^+ g_k^- (g_l^+ e^{\beta B_k} + g_l^- e^{-\beta B_k}) + g_l^+ g_l^- (g_k^+ e^{\beta B_l} + g_k^- e^{-\beta B_l})}{g_k^+ g_k^- + g_l^+ g_l^- + g_k^+ g_l^+ e^{\beta(B_k+B_l)} + g_k^- g_l^- e^{-\beta(B_k+B_l)}} \quad (71)$$

in terms of the operators

$$\begin{aligned} g_k^\pm &\equiv G_k(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, S_l = \pm, \dots, \sigma_N^z) \\ g_l^\pm &\equiv G_l(\sigma_1^z, \dots, S_k = \pm, \dots, \sigma_{l-1}^z, \sigma_{l+1}^z, \dots, \sigma_N^z) \\ B_k &\equiv \sum_{i \neq l} J_{ki} \sigma_i^z \\ B_l &\equiv \sum_{i \neq k} J_{li} \sigma_i^z \end{aligned} \quad (72)$$

6. Final State of the RG procedure

For a system of N spins, we will obtain after $(N-1)$ RG steps a single renormalized spin $S_R = \pm 1$ representing the two ground states of the whole sample, where all spins take the same value S_R . Since there is no renormalized external field left $B_R = 0$, the final effective Hamiltonian for the single spin S_R simply reads

$$\mathcal{H}_N^{final} = G^{final}(N) (1 - \sigma_R^x) \quad (73)$$

where $G^{final}(N)$ is a numerical amplitude. The quantum ground state of zero energy is the symmetric combination of the two classical ferromagnetic ground states as it should

$$|\psi_0^{(final)}\rangle = \frac{|S_R = +\rangle + |S_R = -\rangle}{\sqrt{2}} \quad (74)$$

whereas the excited quantum eigenstate is the antisymmetric combination of the two classical ferromagnetic ground states

$$|\psi_1^{(final)}\rangle = \frac{|S_R = +\rangle - |S_R = -\rangle}{\sqrt{2}} \quad (75)$$

with the eigenvalue

$$E_1^{final}(N) = 2G^{final}(N) \quad (76)$$

The conclusion is that the equilibrium time of Eq. 19 can be thus obtained from the final renormalization amplitude G_N^{final} as

$$t_{eq}(N) \equiv \frac{1}{E_1^{final}(N)} = \frac{1}{2G^{final}(N)} \quad (77)$$

In summary, we have thus defined a renormalized procedure for the amplitudes G that allows to obtain in the end the equilibration time $t_{eq}(N)$. Now to better understand the meaning of the RG rules of Eq. 71, let us give some explicit examples.

7. *Example : first renormalization step starting from the simple Hamiltonian of Eq. 25*

The quantum Hamiltonian of Eq. 25 corresponds to the initial simple case where all amplitudes are unity

$$G_k^{ini}(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) = 1 \quad (78)$$

The first RG step where k and l are grouped into a single renormalized cluster yields the following amplitude (Eq 71 with $g_k^\pm = 1 = g_l^\pm$)

$$\begin{aligned} G_{kl}^R &= e^{-\beta J_{kl}} \frac{\cosh(\beta B_k) + \cosh(\beta B_l)}{1 + \cosh(\beta(B_k + B_l))} \\ &= e^{-\beta J_{kl}} \frac{\cosh(\beta \sum_{i \neq l} J_{ki} \sigma_i^z) + \cosh(\beta \sum_{i \neq k} J_{li} \sigma_i^z)}{1 + \cosh(\beta \sum_{i \neq (k,l)} (J_{ki} + J_{li}) \sigma_i^z)} \end{aligned} \quad (79)$$

So this renormalized amplitude does not remain a number as in the initial condition of Eq 73, but becomes an operator that involves the neighboring spins of k and l .

8. *Example : first renormalization step starting from the Glauber Hamiltonian of Eq. 27*

The quantum Hamiltonian of Eq. 27 corresponds to the initial case

$$G_k^{ini}(\sigma_1^z, \dots, \sigma_{k-1}^z, \sigma_{k+1}^z, \dots, \sigma_N^z) = \frac{1}{2 \cosh(\beta \sum_i J_{ki} \sigma_i^z)} \quad (80)$$

so that the operators of Eq. 72 reads

$$\begin{aligned} g_k^\pm &= \frac{1}{2 \cosh \beta(B_k \pm J_{kl})} \\ g_l^\pm &= \frac{1}{2 \cosh \beta(B_l \pm J_{kl})} \end{aligned} \quad (81)$$

Eq 71 for the renormalized amplitude then becomes

$$\begin{aligned} G_{kl}^R &= \frac{1}{2 \cosh \beta(B_k + B_l) + 2e^{2J_{kl} + B_k - B_l}} + \frac{1}{2 \cosh \beta(B_k + B_l) + 2e^{2J_{kl} - B_k + B_l}} \\ &= \frac{1}{2 \cosh(\beta \sum_{i \neq (k,l)} (J_{ki} + J_{li}) \sigma_i^z) + 2e^{2J_{kl} + \sum_{i \neq (k,l)} (J_{ki} - J_{li}) \sigma_i^z}} \\ &\quad + \frac{1}{2 \cosh(\beta \sum_{i \neq (k,l)} (J_{ki} + J_{li}) \sigma_i^z) + 2e^{2J_{kl} - \sum_{i \neq (k,l)} (J_{ki} - J_{li}) \sigma_i^z}} \end{aligned} \quad (82)$$

So this renormalized amplitude does not remain of the Glauber form of Eq. 80 in terms of the renormalized local field $B_R = B_k + B_l$. Actually it does not even remain a function of the single renormalized local field B_R .

9. *Discussion*

These two examples show that the renormalized amplitudes G_R become generically a function of the σ^z operators of the neighboring spins, ie of the form of Eq. 33. It is not clear to us at this stage how to determine the operator form that would remain stable upon the general RG rule of Eq. 71. In the following, we will thus concentrate on two geometries where we can obtain closed RG rules, namely the one-dimensional case and the Cayley tree.

V. BOUNDARY RENORMALIZATION FOR THE RANDOM FERROMAGNETIC CHAIN

In this section, we consider the random ferromagnetic chain of N spins with the classical energy (Eq 20)

$$U(S_1, \dots, S_N) = - \sum_{i=1}^{N-1} J_{i,i+1} S_i S_{i+1} \quad (83)$$

with free boundary conditions for the two boundary spins S_1 and S_N . The couplings $J_{i,i+1}$ are positive random variables.

A. Closed RG for the simple Hamiltonian of Eq 25 in one dimension

1. First step of the RG rules in the bulk

The first RG step where the two neighboring sites $(k, k+1)$ are grouped into a single ferromagnetic cluster leads to the following renormalized amplitude (Eq 79) for $1 < k < N-1$

$$G_{k,k+1}^R = e^{-\beta J_{k,k+1}} \frac{\cosh(\beta J_{k-1,k}) + \cosh(\beta J_{k+1,k+2})}{1 + \cosh(\beta [J_{k-1,k}\sigma_{k-1}^z + J_{k+1,k+2}\sigma_{k+2}^z])} \quad (84)$$

Since there remains the renormalized local field operator $[J_{k-1,k}\sigma_{k-1}^z + J_{k+1,k+2}\sigma_{k+2}^z]$ in the denominator, this renormalized amplitude is not just a number as the initial condition. To avoid this difficulty, we will now consider what happens near one boundary.

2. First step of the RG rules near the boundary

We now consider the case $k=1$, where we make a ferromagnetic cluster out of the two sites $(1, 2)$ near the boundary. Since the spin zero does not exist, we have $J_{k-1,k} = J_{0,1} = 0$, so that Eq. 84 reduces to a renormalized number (without any operator anymore)

$$G_{1,2}^R = e^{-\beta J_{1,2}} \frac{1 + \cosh(\beta J_{2,3})}{1 + \cosh(\beta J_{2,3})} = e^{-\beta J_{1,2}} \quad (85)$$

This shows that we may define a simple closed boundary RG procedure as follows.

3. Closed Boundary RG procedure

If we iterate the renormalization near the boundary, the renormalized state after $(n-1)$ RG steps will be the following : the spins $(1, 2, \dots, n)$ have been grouped together into a single renormalized spin with some associated renormalized amplitude $G_{[1,n]}^R$. The other spins $(n+1, \dots, N)$ of the chain are still in their initial form with amplitude unity $G_k = 1$ for $n+1 \leq k \leq N$. Let us now perform the n RG step where the spin $(n+1)$ is included in the boundary cluster. The RG rule of Eq. 71 yields the recurrence

$$G_{[1,n+1]}^R = e^{-\beta J_{n,n+1}} G_{[1,n]}^R \left[\frac{2G_{[1,n]}^R + 2 \cosh(\beta J_{n+1,n+2})}{(G_{[1,n]}^R)^2 + 1 + G_{[1,n]}^R (2 \cosh(\beta J_{n+1,n+2}))} \right] \quad (86)$$

Within the *bulk*, i.e. for $J_{n+1,n+2} > 0$, in order to be consistent with our previous approximations in the low-temperature limit (Eq 1), we may replace

$$2 \cosh(\beta J_{n+1,n+2}) \underset{\beta J_{n+1,n+2} \gg 1}{\simeq} e^{\beta J_{n+1,n+2}} \quad (87)$$

Moreover, we expect that the renormalized amplitudes G^R only decay upon renormalization, and are thus smaller than their initial unity value (see already the first step of Eq. 85)

$$G_{[1,n]}^R \leq 1 \quad (88)$$

So Eq. 86 reduces to

$$G_{[1,n+1]}^R \simeq e^{-\beta J_{n,n+1}} G_{[1,n]}^R \left[\frac{e^{\beta J_{n+1,n+2}}}{1 + G_{[1,n]}^R e^{\beta J_{n+1,n+2}}} \right] \quad (89)$$

i.e. this recurrence becomes simpler in terms of inverse variables

$$\frac{e^{\beta J_{n+1,n+2}}}{G_{[1,n+1]}^R} \simeq e^{\beta J_{n,n+1} + \beta J_{n+1,n+2}} + \frac{e^{\beta J_{n,n+1}}}{G_{[1,n]}^R} \quad (90)$$

4. Final result for a chain of N spins

For a finite chain of N spins where $J_{N,N+1} = 0$, the recurrence of Eq. 86 yields for the last step with $n+1 = N$

$$\begin{aligned} G_{[1,N]}^{Rfinal} &= e^{-\beta J_{N-1,N}} G_{[1,N-1]}^R \left[\frac{2G_{[1,N-1]}^R + 2}{(G_{[1,N-1]}^R)^2 + 1 + G_{[1,N-1]}^R} \right] \\ &= e^{-\beta J_{N-1,N}} G_{[1,N-1]}^R \left[\frac{2}{1 + G_{[1,N-1]}^R} \right] \end{aligned} \quad (91)$$

i.e. using inverse variables

$$\frac{1}{G_{[1,N]}^{Rfinal}} = \frac{1}{2} \left[e^{\beta J_{N-1,N}} + \frac{e^{\beta J_{N-1,N}}}{G_{[1,N-1]}^R} \right] \quad (92)$$

We now use iteratively the recurrence of Eq. 90 valid in the bulk to obtain

$$\begin{aligned} \frac{1}{G_{[1,N]}^R} &= \frac{1}{2} \left[e^{\beta J_{N-1,N}} + e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}} + \frac{e^{\beta J_{N-2,N-1}}}{G_{[1,N-2]}^R} \right] \\ &= \frac{1}{2} \left[e^{\beta J_{N-1,N}} + e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}} + e^{\beta J_{N-3,N-2} + \beta J_{N-2,N-1}} + \frac{e^{\beta J_{N-3,N-2}}}{G_{[1,N-3]}^R} \right] \\ &\simeq \frac{1}{2} \sum_{k=1}^N e^{\beta(J_{k-1,k} + J_{k,k+1})} \end{aligned} \quad (93)$$

(with the notations $J_{0,1} = 0 = J_{N,N+1}$)

The conclusion of this RG procedure is thus that the equilibration time of the finite random chain of N spins reads for the 'simple' dynamics (Eq. 77)

$$t_{eq}^{simple}(N) = \frac{1}{E_1^{final}(N)} = \frac{1}{2G_{[1,N]}^R} = \frac{1}{4} \sum_{k=1}^N e^{\beta(J_{k-1,k} + J_{k,k+1})} \quad (94)$$

B. Closed RG for dynamics depending only on the local field in one dimension

We now consider the more general case where the amplitude G_k are a single even function $G(x) = G(-x)$ of the local field (see Eq. 34).

$$G_k = G(J_{k-1,k} \sigma_{k-1}^z + J_{k,k+1} \sigma_{k+1}^z) \quad (95)$$

The Glauber Hamiltonian of Eq 27 corresponds to the special function

$$G(x) = \frac{1}{2 \cosh(\beta x)} \quad (96)$$

1. Closed Boundary RG procedure

If we iterate the renormalization near the boundary as in section V A 3, we obtain finally the following recurrence (Eq 71) :

$$G_{[1,n+1]}^R = e^{-\beta J_{n,n+1}} G_{[1,n]}^R \frac{G_{[1,n]}^R (f_{n+1}^+ + f_{n+1}^-) + f_{n+1}^+ f_{n+1}^- (2 \cosh(\beta J_{n+1,n+2}))}{(G_{[1,n]}^R)^2 + f_{n+1}^+ f_{n+1}^- + G_{[1,n]}^R (f_{n+1}^+ e^{\beta J_{n+1,n+2}} + f_{n+1}^- e^{-\beta J_{n+1,n+2}})} \quad (97)$$

in terms of the numbers

$$f_{n+1}^\pm \equiv G(J_{n,n+1} \pm J_{n+1,n+2}) \quad (98)$$

Within the *bulk*, i.e. for $J_{n+1,n+2} > 0$, in order to be consistent with our previous approximations in the low-temperature limit (Eq 1), we may use Eqs 87 and the fact that the renormalized amplitudes G^R only decay upon renormalization to simplify Eq 97 into

$$G_{[1,n+1]}^R \simeq e^{-\beta J_{n,n+1}} G_{[1,n]}^R \frac{f_{n+1}^+ f_{n+1}^- e^{\beta J_{n+1,n+2}}}{f_{n+1}^+ f_{n+1}^- + G_{[1,n]}^R f_{n+1}^+ e^{\beta J_{n+1,n+2}}} \quad (99)$$

i.e. this recurrence becomes simpler in terms of inverse variables

$$\frac{e^{\beta J_{n+1,n+2}}}{G_{[1,n+1]}^R} \simeq \frac{e^{\beta J_{n,n+1} + \beta J_{n+1,n+2}}}{f_{n+1}^-} + \frac{e^{\beta J_{n,n+1}}}{G_{[1,n]}^R} \quad (100)$$

2. Final result for a chain of N spins

For a finite chain of N spins where $J_{N,N+1} = 0$, Eq 98 reads

$$f_N^\pm \equiv G(J_{N-1,N} \pm 0) = f_N^- \quad (101)$$

and the recurrence of Eq. 97 yields for the last step with $n+1 = N$

$$\begin{aligned} G_{[1,N]}^{Rfinal} &= e^{-\beta J_{N-1,N}} G_{[1,N-1]}^R \frac{2f_N^-(G_{[1,N-1]}^R + f_N^-)}{(G_{[1,N-1]}^R + f_N^-)^2} \\ &= e^{-\beta J_{N-1,N}} G_{[1,N-1]}^R \frac{2f_N^-}{G_{[1,N-1]}^R + f_N^-} \end{aligned} \quad (102)$$

i.e. using inverse variables

$$\frac{1}{G_{[1,N]}^{Rfinal}} = \frac{1}{2} \left[\frac{e^{\beta J_{N-1,N}}}{f_N^-} + \frac{e^{\beta J_{N-1,N}}}{G_{[1,N-1]}^R} \right] \quad (103)$$

We now use iteratively the recurrence of Eq. 100 valid in the bulk to obtain

$$\begin{aligned} \frac{1}{G_{[1,N]}^R} &= \frac{1}{2} \left[e^{\beta J_{N-1,N}} + \frac{e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}}}{f_{N-1}^-} + \frac{e^{\beta J_{N-2,N-1}}}{G_{[1,N-2]}^R} \right] \\ &= \frac{1}{2} \left[\frac{e^{\beta J_{N-1,N}}}{f_N^-} + \frac{e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}}}{f_{N-1}^-} + \frac{e^{\beta J_{N-3,N-2} + \beta J_{N-2,N-1}}}{f_{N-2}^-} + \frac{e^{\beta J_{N-3,N-2}}}{G_{[1,N-3]}^R} \right] \\ &\simeq \frac{1}{2} \sum_{k=1}^N \frac{e^{\beta(J_{k-1,k} + J_{k,k+1})}}{f_k^-} \end{aligned} \quad (104)$$

(with the notations $J_{0,1} = 0 = J_{N,N+1}$)

Using Eq. 98, the conclusion of this RG procedure is thus that the equilibration time of the finite random chain of N spins reads for the dynamics defined by the amplitudes of Eq. 97

$$t_{eq}(N) = \frac{1}{E_1^{final}(N)} = \frac{1}{2G_{[1,N]}^R} = \frac{1}{4} \sum_{k=1}^N \frac{e^{\beta(J_{k-1,k} + J_{k,k+1})}}{G(J_{k-1,k} - J_{k,k+1})} \quad (105)$$

In particular for the Glauber dynamics corresponding to Eq. 96, the equilibration time reads

$$\begin{aligned} t_{eq}^{glauber}(N) &= \frac{1}{4} \sum_{k=1}^N e^{\beta(J_{k-1,k} + J_{k,k+1})} 2 \cosh(\beta J_{k-1,k} - \beta J_{k,k+1}) = \frac{1}{4} \sum_{k=1}^N [e^{2\beta J_{k-1,k}} + e^{2\beta J_{k,k+1}}] \\ &= \frac{1}{2} \left[1 + \sum_{k=1}^{N-1} e^{2\beta J_{k,k+1}} \right] \end{aligned} \quad (106)$$

C. Discussion

In this section, we have obtained a closed RG procedure in dimension $d = 1$ that yields the explicit result of Eq. 105 for the equilibration time of a finite chain. The validity of this RG result is checked in Appendix A using another method. Since we compare the explicit expression in terms of the set of all random couplings, for various choices of the transition rates, this agreement shows the exactness of the RG procedure near zero temperature.

The physical meaning of Eq. 105 is that the equilibration time is the sum of N random variables, possibly slightly correlated (the same random coupling $J_{n,n+1}$ appears in the two terms corresponding to $k = n$ and $k = n + 1$). For the Glauber case of Eq. 106, even these slight correlations disappear and the equilibration time reduces to the sum of independent random variables $e^{2\beta J_{k,k+1}}$ whose distribution can be computed from the distribution of the random couplings $J_{k,k+1}$.

Note that in the limit of a pure chain where all ferromagnetic coupling have the same value J , Eq 105 reduces to

$$t_{eq}^{pure}(N) = \frac{1}{4} \left[2 \frac{e^{\beta J}}{G(J)} + (N-2) \frac{e^{2\beta J}}{G(0)} \right] \propto \frac{N e^{2\beta J}}{G(0)} \quad (107)$$

as expected : the Arrhenius factor $e^{2\beta J}$ comes from the barrier ($2J$) to create a domain-wall at one boundary, and the prefactor N comes from the small probability of order $1/N$ that this domain-wall crosses the whole system of size N instead of returning back. For the Glauber case where $G(0) = 2$, Eq. 107 is in agreement with the leading term near zero temperature of the open pure chain discussed in Refs [24].

VI. BOUNDARY RENORMALIZATION FOR THE ISING MODEL ON THE CAYLEY TREE

In this section, we consider the pure ferromagnetic Ising model with the classical energy of Eq 20 defined on Cayley tree of branching ratio K (coordination $(K + 1)$) with N generations, and with free boundary conditions on all the boundary spins. We focus on the dynamics corresponding to the simplest choice of Eq. 24, where the corresponding quantum Hamiltonian of Eq. 25 has initial amplitude G_k all equal to unity.

We wish to define a closed boundary RG procedure that preserves the symmetry between the K offsprings of a given site. So the basic RG step concerns K renormalized boundary spins (S_1, S_2, \dots, S_K) whose renormalized dynamics is described by some renormalized amplitude G (which is a number and not an operator) and their common ancestor spin S whose dynamics is still described by the initial amplitude $g = 1$ and by the external field $B_a = J_a \sigma_a^z$ induced by its next ancestor spin S_a . The ferromagnetic couplings have still their initial value J so that we have to study the following effective Hamiltonian for these $(K + 1)$ spins (S_1, \dots, S_K, S)

$$H_{K+1}^{simple} \equiv \left(e^{-\beta \sigma^z (\sum_{i=1}^K J \sigma_i^z + B_a)} - \sigma^x \right) + G \sum_{i=1}^K \left(e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^x \right) \quad (108)$$

After the first RG step, the renormalized amplitude G is expected to become smaller and smaller, so the most appropriate approach is a perturbative analysis in the parameter G .

A. Properties of the Hamiltonian H_{K+1}^{simple} for $G = 0$

For $G = 0$, the spins (S_1, \dots, S_K) cannot flip and are thus frozen, so the problem reduces to the single spin S in the external field $B = \left(\sum_{i=1}^K J S_i + B_a \right)$ that we have already studied in section IV B 1. So the 2^{K+1} states can be decomposed into

(i) the 2^K states corresponding to Eq. 40

$$|v_0^{S_1, S_2, \dots, S_K} > \equiv \prod_{j=1}^K |S_j > \sum_{S=\pm} \frac{e^{\frac{\beta}{2} S (\sum_{i=1}^K J S_i + B_a)}}{\sqrt{2 \cosh \beta (\sum_{i=1}^K J S_i + B_a)}} |S > \quad (109)$$

that have a vanishing eigenvalue for $G = 0$

$$\left(e^{-\beta \sigma^z (\sum_{i=1}^K J \sigma_i^z + B_a)} - \sigma^x \right) |v_0^{S_1, S_2, \dots, S_K} > = 0 \quad (110)$$

The physical interpretation is that the spin S is at equilibrium with respect to the frozen spins (S_1, \dots, S_K) .

(ii) the 2^K states corresponding to Eq. 41

$$|w_0^{S_1, S_2, \dots, S_K} \rangle \equiv \prod_{j=1}^K |S_j \rangle \sum_{S=\pm} \frac{S e^{-\frac{\beta}{2} S (\sum_{i=1}^K J S_i + B_a)}}{\sqrt{2 \cosh \beta (\sum_{i=1}^K J S_i + B_a)}} |S \rangle \quad (111)$$

that have a finite eigenvalue for $G = 0$

$$\left(e^{-\beta \sigma^z (\sum_{i=1}^K J \sigma_i^z + B_a)} - \sigma^x \right) |w_0^{S_1, S_2, \dots, S_K} \rangle = \left[2 \cosh \beta \left(\sum_{i=1}^K J S_i + B_a \right) \right] |w_0^{S_1, S_2, \dots, S_K} \rangle \quad (112)$$

Note that the exact ground state (Eq 16) of the Hamiltonian of Eq. 108

$$\begin{aligned} |\psi_0 \rangle &= \frac{1}{\sqrt{Z_{K+1}}} \sum_{S_1=\pm} \sum_{S_2=\pm} \dots \sum_{S_K=\pm} e^{\frac{\beta}{2} S (\sum_{i=1}^K J S_i + B_a)} |S_1 \rangle |S_2 \rangle \dots |S_K \rangle |S \rangle \\ Z_{K+1} &\equiv \sum_{S_1=\pm} \sum_{S_2=\pm} \dots \sum_{S_K=\pm} e^{\beta S (\sum_{i=1}^K J S_i + B_a)} = \sum_{S_1=\pm} \sum_{S_2=\pm} \dots \sum_{S_K=\pm} \left[2 \cosh \beta \left(\sum_{i=1}^K J S_i + B_a \right) \right] \end{aligned} \quad (113)$$

belongs to the subspace spanned by the 2^K states $|v_0^{S_1, S_2, \dots, S_K} \rangle$ (Eq 110)

$$|\psi_0 \rangle = \frac{1}{\sqrt{Z_{K+1}}} \sum_{S_1=\pm} \sum_{S_2=\pm} \dots \sum_{S_K=\pm} \sqrt{2 \cosh \beta \left(\sum_{i=1}^K J S_i + B_a \right)} |v_0^{S_1, S_2, \dots, S_K} \rangle \quad (114)$$

B. Perturbation in the parameter G

We have seen that for $G = 0$, there are 2^K states (Eq 110) that have zero energy. For small $G > 0$, the perturbation will lift this degeneracy : the exact ground state of Eq. 114 will keep its zero energy for arbitrary G , but the other $(2^K - 1)$ eigenvalues will become positive as soon as $G > 0$. To determine them, we need to diagonalize the perturbation within the subspace spanned by the 2^K vectors $|v_0^{S_1, S_2, \dots, S_K} \rangle$, i.e. we look for an eigenstate via the linear combination

$$|u_\lambda \rangle = \sum_{S_1=\pm, S_2=\pm, \dots, S_K=\pm} T^{S_1, S_2, \dots, S_K} |v_0^{S_1, S_2, \dots, S_K} \rangle \quad (115)$$

The eigenvalue-equation

$$\begin{aligned} 0 &= (H_{K+1}^{simple} - \lambda) |u_\lambda \rangle \\ &= \sum_{S_1=\pm, S_2=\pm, \dots, S_K=\pm} T^{S_1, S_2, \dots, S_K} \left[\sum_{i=1}^K G \left(e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^x \right) - \lambda \right] |v_0^{S_1, S_2, \dots, S_K} \rangle \end{aligned} \quad (116)$$

can be projected onto the 2^K bra $\langle v_0^{S'_1, S'_2, \dots, S'_K} |$ to obtain a system of 2^K linear equations for the 2^K coefficients T^{S_1, S_2, \dots, S_K}

$$\begin{aligned} 0 &= \langle v_0^{S'_1, \dots, S'_K} | (H_{K+1}^{simple} - \lambda) |u_\lambda \rangle \\ &= \langle v_0^{S'_1, \dots, S'_K} | \sum_{S_1=\pm, \dots, S_K} T^{S_1, S_2, \dots, S_K} \left[\sum_{i=1}^K G \left(e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^x \right) - \lambda \right] |v_0^{S_1, \dots, S_K} \rangle \end{aligned} \quad (117)$$

From the matrix elements

$$\begin{aligned} \langle v_0^{S'_1, S'_2, \dots, S'_K} | \left[e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^x \right] |v_0^{S_1, S_2, \dots, S_K} \rangle &= \left(\prod_j \delta_{S'_j, S_j} \right) \frac{2 \cosh \beta (\sum_{j \neq i} J S_j + B_a)}{2 \cosh \beta (\sum_{j=1}^K J S_j + B_a)} \\ &- \left(\prod_{j \neq i} \delta_{S'_j, S_j} \right) \delta_{S'_i, -S_i} \frac{2 \cosh \beta (\sum_{j \neq i} J S_j + B_a)}{\sqrt{2 \cosh \beta (\sum_{j \neq i} J S_j - J S_i + B_a)} \sqrt{2 \cosh \beta (\sum_{j \neq i} J S_j + J S_i + B_a)}} \end{aligned} \quad (118)$$

we obtain that Eq 117 yields

$$\begin{aligned}
0 &= \sum_{S_1=\pm, \dots, S_K=\pm} T^{S_1, \dots, S_K} \left[\sum_{i=1}^K G \langle v_0^{S'_1, \dots, S'_K} | (e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^x) | v_0^{S_1, \dots, S_K} \rangle - \lambda \langle v_0^{S'_1, \dots, S'_K} | v_0^{S_1, \dots, S_K} \rangle \right] \\
&= \left[G \sum_{i=1}^K \frac{2 \cosh \beta (\sum_{j \neq i} J S'_j + B_a)}{2 \cosh \beta (\sum_{j=1}^K J S'_j + B_a)} - \lambda \right] T^{S'_1, \dots, S'_K} \\
&\quad - G \sum_{i=1}^K \frac{2 \cosh \beta (\sum_{j \neq i} J S'_j + B_a)}{\sqrt{2 \cosh \beta (\sum_{j \neq i} J S'_j - J S'_i + B_a)} \sqrt{2 \cosh \beta (\sum_{j \neq i} J S'_j + J S'_i + B_a)}} T^{S'_1, \dots, -S'_i, \dots, S'_K}
\end{aligned} \tag{119}$$

Let us now use the symmetry between the K branches to note $t(k)$ the components $T^{S'_1, S'_2, \dots, S'_K}$ where k spins take the value $(+)$ and $(K - k)$ take the value $-$, i.e.

$$T^{S'_1, S'_2, \dots, S'_K} = t \left(k = \sum_{i=1}^K \frac{1 + S'_i}{2} \right) \tag{120}$$

For the two extremal cases $t(K)$ (all spins are $(+)$) and $t(0)$ (all spins are $(-)$), Eq 119 becomes

$$\begin{aligned}
0 &= \left[GK \frac{2 \cosh \beta ((K-1)J + B_a)}{2 \cosh \beta (KJ + B_a)} - \lambda \right] t(K) - GK \frac{2 \cosh \beta ((K-1)J + B_a)}{\sqrt{2 \cosh \beta ((K-2)J + B_a)} \sqrt{2 \cosh \beta (KJ + B_a)}} t(K-1) \\
0 &= \left[GK \frac{2 \cosh \beta ((K-1)J - B_a)}{2 \cosh \beta (KJ - B_a)} - \lambda \right] t(0) - GK \frac{2 \cosh \beta ((K-1)J - B_a)}{\sqrt{2 \cosh \beta ((K-2)J - B_a)} \sqrt{2 \cosh \beta (KJ - B_a)}} t(1)
\end{aligned} \tag{121}$$

whereas for the non-extremal case $0 < k < K$, Eq. 119 reads

$$\begin{aligned}
0 &= \left[Gk \frac{2 \cosh \beta ((2k-K-1)J + B_a)}{2 \cosh \beta ((2k-K)J + B_a)} + G(K-k) \frac{2 \cosh \beta ((2k-K+1)J + B_a)}{2 \cosh \beta ((2k-K)J + B_a)} - \lambda \right] t(k) \\
&\quad - Gk \frac{2 \cosh \beta ((2k-K-1)J + B_a)}{\sqrt{2 \cosh \beta ((2k-K-2)J + B_a)} \sqrt{2 \cosh \beta ((2k-K)J + B_a)}} t(k-1) \\
&\quad - G(K-k) \frac{2 \cosh \beta ((2k-K+1)J + B_a)}{\sqrt{2 \cosh \beta ((2k-K+2)J + B_a)} \sqrt{2 \cosh \beta ((2k-K)J + B_a)}} t(k+1)
\end{aligned} \tag{122}$$

It is easy to check that the components (not normalized here) of the exact ground state of Eq 113

$$t_0(k) = \sqrt{2 \cosh \beta ((2k-K)J + B_a)} \tag{123}$$

satisfy Equations 121 and 122 for $\lambda = 0$ as it should.

To compute the lowest non-vanishing eigenvalue λ_1 , it is consistent to set $\lambda_1 = 0$ in all eqs 122 concerning the non-extremal cases $0 < k < K$, leading to

$$\begin{aligned}
0 &= \left[k \frac{2 \cosh \beta ((2k-K-1)J + B_a)}{\sqrt{2 \cosh \beta ((2k-K)J + B_a)}} + (K-k) \frac{2 \cosh \beta ((2k-K+1)J + B_a)}{\sqrt{2 \cosh \beta ((2k-K)J + B_a)}} \right] t(k) \\
&\quad - k \frac{2 \cosh \beta ((2k-K-1)J + B_a)}{\sqrt{2 \cosh \beta ((2k-K-2)J + B_a)}} t(k-1) - (K-k) \frac{2 \cosh \beta ((2k-K+1)J + B_a)}{\sqrt{2 \cosh \beta ((2k-K+2)J + B_a)}} t(k+1)
\end{aligned} \tag{124}$$

that may be solved in terms of the boundary conditions $t(K)$ and $t(0)$.

To obtain the explicit solution, it is convenient to introduce the amplitudes $A(k)$ with respect to the ground state components of Eq. 123

$$t(k) \equiv A(k) t_0(k) = A(k) \sqrt{2 \cosh \beta ((2k-K)J + B_a)} \tag{125}$$

so that Eq. 124 takes the simpler form

$$A(k) = p_-(k) A(k-1) + p_+(k) A(k+1) \tag{126}$$

with the notations

$$\begin{aligned} p_-(k) &\equiv \frac{k2 \cosh \beta((2k - K - 1) + B_a)}{[k2 \cosh \beta((2k - K - 1) + B_a) + (K - k)2 \cosh \beta((2k - K + 1) + B_a)]} \\ p_+(k) &\equiv \frac{(K - k)2 \cosh \beta((2k - K + 1) + B_a)}{[k2 \cosh \beta((2k - K - 1) + B_a) + (K - k)2 \cosh \beta((2k - K + 1) + B_a)]} = 1 - p_-(k) \end{aligned} \quad (127)$$

Let us introduce two linearly independent solutions. The solution corresponding to the boundary conditions

$$\begin{aligned} Q_K(0) &= 0 \\ Q_K(K) &= 1 \end{aligned} \quad (128)$$

can be obtained by recurrence [25] and reads

$$Q_K(k) = \frac{R_K(1, k)}{R_K(1, K)} \quad (129)$$

using Kesten variables [26]

$$\begin{aligned} R_K(1, 0) &= 0 \\ R_K(1, 1) &= 1 \\ R_K(1, k \geq 2) &= 1 + \sum_{m=1}^{k-1} \prod_{n=1}^m \frac{p_-(n)}{p_+(n)} \\ R_K(1, K) &= 1 + \sum_{m=1}^{K-1} \prod_{n=1}^m \frac{p_-(n)}{p_+(n)} = 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \dots + \frac{p_-(1)p_-(2)\dots p_-(K-1)}{p_+(1)p_+(2)\dots p_+(K-1)} \end{aligned} \quad (130)$$

Similarly, the solution corresponding to the boundary conditions

$$\begin{aligned} Q_0(0) &= 1 \\ Q_0(K) &= 0 \end{aligned} \quad (131)$$

reads

$$Q_0(k) = \frac{R_0(k, K-1)}{R_0(0, K-1)} \quad (132)$$

with

$$\begin{aligned} R_0(K, K-1) &= 0 \\ R_0(K-1, K-1) &= 1 \\ R_0(k \leq K-2, K-1) &= 1 + \sum_{m=k+1}^{K-1} \prod_{n=m}^{K-1} \frac{p_+(n)}{p_-(n)} \\ R_0(0, K-1) &= 1 + \sum_{m=1}^{K-1} \prod_{n=m}^{K-1} \frac{p_+(n)}{p_-(n)} = 1 + \frac{p_+(K-1)}{p_-(K-1)} + \dots + \frac{p_+(K-1)p_+(K-2)\dots p_+(1)}{p_-(K-1)p_-(K-2)\dots p_-(1)} \end{aligned} \quad (133)$$

It is useful to introduce the continuation of the ground state components of Eq 123 to half-integers to rewrite the ratios

$$\frac{p_-(k)}{p_+(k)} = \frac{k2 \cosh \beta((2k - K - 1) + B_a)}{(K - k)2 \cosh \beta((2k - K + 1) + B_a)} = \frac{kt_0^2(k - \frac{1}{2})}{(K - k)t_0^2(k + \frac{1}{2})} \quad (134)$$

and the products

$$\begin{aligned} \prod_{n=1}^m \frac{p_-(n)}{p_+(n)} &= \prod_{n=1}^m \left[\frac{nt_0^2(n - \frac{1}{2})}{(K - n)t_0^2(n + \frac{1}{2})} \right] = \frac{m!(K-1-m)!}{(K-1)!} \frac{t_0^2(\frac{1}{2})}{t_0^2(m + \frac{1}{2})} \\ \prod_{n=m}^{K-1} \frac{p_+(n)}{p_-(n)} &= \prod_{n=m}^{K-1} \frac{(K-n)t_0^2(n + \frac{1}{2})}{nt_0^2(n - \frac{1}{2})} = \frac{(m-1)!(K-m)!}{(K-1)!} \frac{t_0^2(K - \frac{1}{2})}{t_0^2(m - \frac{1}{2})} \end{aligned} \quad (135)$$

In particular in the following, we will need the two denominators

$$R_K(1, K) = 1 + \sum_{m=1}^{K-1} \frac{m!(K-1-m)!}{(K-1)!} \frac{t_0^2(\frac{1}{2})}{t_0^2(m+\frac{1}{2})} = t_0^2\left(\frac{1}{2}\right) \sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m t_0^2(m+\frac{1}{2})} \quad (136)$$

and

$$R_0(0, K-1) = 1 + \sum_{m=1}^{K-1} \frac{(m-1)!(K-m)!}{(K-1)!} \frac{t_0^2(K-\frac{1}{2})}{t_0^2(m-\frac{1}{2})} = t_0^2\left(K-\frac{1}{2}\right) \sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m t_0^2(m+\frac{1}{2})} \quad (137)$$

that determine the solutions near the boundaries of Q_K

$$\begin{aligned} Q_K(1) &= \frac{R_K(1, 1)}{R_K(1, K)} = \frac{1}{R_K(1, K)} \\ 1 - Q_K(K-1) &= \frac{R_K(1, K) - R_K(1, K-1)}{R_K(1, K)} = \frac{1}{R_0(0, K-1)} \end{aligned} \quad (138)$$

and of Q_0

$$\begin{aligned} Q_0(K-1) &= \frac{R_0(K-1, K-1)}{R_0(0, K-1)} = \frac{1}{R_0(0, K-1)} \\ 1 - Q_0(1) &= \frac{R_0(0, K-1) - R_0(1, K-1)}{R_0(0, K-1)} = \frac{1}{R_K(1, K)} \end{aligned} \quad (139)$$

The solution of the system 126 that satisfy the boundary conditions (Eq 125)

$$\begin{aligned} A(0) &= \frac{t(0)}{t_0(0)} \\ A(K) &= \frac{t(K)}{t_0(K)} \end{aligned} \quad (140)$$

can be obtained by the linear combination

$$A(k) = A(0)Q_0(k) + A(K)Q_K(k) = \frac{t(0)}{t_0(0)}Q_0(k) + \frac{t(K)}{t_0(K)}Q_K(k) \quad (141)$$

so that the solution of the system 124 reads

$$t(k) = A(k)t_0(k) = \left[\frac{t(0)}{t_0(0)}Q_0(k) + \frac{t(K)}{t_0(K)}Q_K(k) \right] t_0(k) \quad (142)$$

To determine λ_1 , we just need to replace

$$\begin{aligned} t(1) &= \left[\frac{t_0(1)}{t_0(0)}Q_0(1) \right] t(0) + \left[\frac{t_0(1)}{t_0(K)}Q_K(1) \right] t(K) \\ t(K-1) &= \left[\frac{t_0(K-1)}{t_0(0)}Q_0(K-1) \right] t(0) + \left[\frac{t_0(K-1)}{t_0(K)}Q_K(K-1) \right] t(K) \end{aligned} \quad (143)$$

in Eqs 121 to obtain the following system of two linear equations for the two components $t_{\lambda_1}(0)$ and $t_{\lambda_1}(K)$

$$\begin{aligned} 0 &= \left[1 - Q_K(K-1) - \lambda_1 \frac{t_0^2(K)}{GK t_0^2(K-\frac{1}{2})} \right] t_{\lambda_1}(K) - \frac{t_0(K)}{t_0(0)}Q_0(K-1)t_{\lambda_1}(0) \\ 0 &= \left[1 - Q_0(1) - \lambda_1 \frac{t_0^2(0)}{GK t_0^2(\frac{1}{2})} \right] t_{\lambda_1}(0) - \frac{t_0(0)}{t_0(K)}Q_K(1)t_{\lambda_1}(K) \end{aligned} \quad (144)$$

The two components (not normalized here) are orthogonal to Eq 123 as it should and read

$$\begin{aligned} t_{\lambda_1}(0) &= t_0(K) = \sqrt{2 \cosh \beta(KJ + B_a)} \\ t_{\lambda_1}(K) &= -t_0(0) = -\sqrt{2 \cosh \beta(KJ - B_a)} \end{aligned} \quad (145)$$

The corresponding eigenvalue λ_1 reads using Eqs 136, 137, 138, Eq 139

$$\lambda_1 = \frac{GK}{\sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta((2m+1-K)J+B_a)}} \left[\frac{1}{2 \cosh \beta(KJ-B_a)} + \frac{1}{2 \cosh \beta(KJ+B_a)} \right] \quad (146)$$

Since this expression is unchanged via the transformation $B_a = JS_a \rightarrow -B_a$, we may replace B_a by its absolute value $|B_a| = J$ to obtain the final expression for the lowest non-vanishing eigenvalue λ_1 at first order in perturbation with respect to the parameter G

$$\lambda_1 = \frac{GK}{\sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J(2m+2-K)}} \left[\frac{1}{2 \cosh \beta J(K-1)} + \frac{1}{2 \cosh \beta J(K+1)} \right] + O(G^2) \quad (147)$$

C. Renormalization rule for the amplitude G^R

Let us now project the Hamiltonian of Eq. 108 onto its two lowest eigenvalues $\lambda_0 = 0$ and λ_1

$$H_{K+1}^{simple} \simeq \frac{\lambda_1}{\langle u_{\lambda_1} | u_{\lambda_1} \rangle} |u_{\lambda_1} \rangle \langle u_{\lambda_1}| \quad (148)$$

where the eigenvector of Eq. 115 can be approximated at low temperature by its two components onto fully ferromagnetic states

$$\begin{aligned} |u_{\lambda_1} \rangle &\simeq t_{\lambda_1}(K) \left(\prod_{j=1}^K |S_j = 1 \rangle \right) \sum_{S=\pm} \frac{e^{\frac{\beta}{2} S(KJ+B_a)}}{\sqrt{2 \cosh \beta(KJ+B_a)}} |S \rangle \\ &+ t_{\lambda_1}(0) \left(\prod_{j=1}^K |S_j = -1 \rangle \right) \sum_{S=\pm} \frac{e^{\frac{\beta}{2} S(-KJ+J_a)}}{\sqrt{2 \cosh \beta(KJ-B_a)}} |S \rangle \\ &\simeq t_{\lambda_1}(K) \left(\prod_{j=1}^K |S_j = 1 \rangle \right) |S = +1 \rangle + t_{\lambda_1}(0) \left(\prod_{j=1}^K |S_j = -1 \rangle \right) |S = -1 \rangle \end{aligned} \quad (149)$$

with the coefficients (Eq 145)

$$\begin{aligned} t_{\lambda_1}(0) &= \sqrt{2 \cosh \beta(KJ+B_a)} \simeq e^{\frac{\beta}{2}(KJ+B_a)} \\ t_{\lambda_1}(K) &= -\sqrt{2 \cosh \beta(KJ-B_a)} \simeq -e^{\frac{\beta}{2}(KJ-B_a)} \end{aligned} \quad (150)$$

Finally at leading order near zero temperature, one obtains

$$|u_{\lambda_1} \rangle \simeq e^{\frac{\beta}{2} KJ} \left[e^{\frac{\beta}{2} B_a} \left(\prod_{j=1}^K |S_j = 1 \rangle \right) |S = +1 \rangle - e^{-\frac{\beta}{2} B_a} \left(\prod_{j=1}^K |S_j = -1 \rangle \right) |S = -1 \rangle \right] \quad (151)$$

with the corresponding normalization (using $B_a = JS_a$)

$$\langle u_{\lambda_1} | u_{\lambda_1} \rangle \simeq e^{\beta(KJ+B_a)} + e^{\beta(KJ-B_a)} = e^{\beta(KJ)} 2 \cosh(\beta B_a) = e^{\beta(KJ)} 2 \cosh(\beta J) \quad (152)$$

In terms of the renormalized spin

$$\begin{aligned} |S_R = + \rangle &\equiv \left(\prod_{j=1}^K |S_j = 1 \rangle \right) |S = + \rangle \\ |S_R = - \rangle &\equiv \left(\prod_{j=1}^K |S_j = -1 \rangle \right) |S = - \rangle \end{aligned} \quad (153)$$

and of the external local field $B_R = B_a = JS_a$, the effective Hamiltonian of Eq. 148 can be rewritten as an elementary operator (Eq. 44)

$$H_{K+1}^{simple} \simeq G_R \left(e^{-\frac{\beta}{2} B_R} |S_R = + \rangle - e^{\frac{\beta}{2} B_R} |S_R = - \rangle \right) \left(e^{-\frac{\beta}{2} B_R} \langle S_R = +| - e^{\frac{\beta}{2} B_R} \langle S_R = -| \right) \quad (154)$$

with the renormalized amplitude (using Eq 147)

$$G_R = \frac{\lambda_1}{2 \cosh \beta J} = \frac{GK}{2 \cosh \beta J \sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J (2m+2-K)}} \left[\frac{1}{2 \cosh \beta J (K-1)} + \frac{1}{2 \cosh \beta J (K+1)} \right] \quad (155)$$

To be consistent with the previous low-temperature approximations, we now should evaluate the leading behavior of Eq. 155 near zero temperature, i.e. we should replace hyperbolic functions by exponentials. In particular, one has

$$\begin{aligned} \frac{1}{2 \cosh \beta J (2m+2-K)} &= \frac{1}{e^{\beta J (2m+2-K)} + e^{-\beta J (2m+2-K)}} \simeq \frac{1}{2} \text{ if } m = \frac{K}{2} - 1 \\ &\simeq e^{-\beta J |2m+2-K|} \text{ if } m \neq \frac{K}{2} - 1 \end{aligned} \quad (156)$$

so that the leading term near low temperature of Eq. 155 depends on the parity of K .

1. Leading behavior near zero temperature for even K

When the branching ratio K is even, then $(\frac{K}{2} - 1)$ is an integer, so that the integer m can take this value, and the sum in the denominator of Eq. 155 is dominated by this contribution

$$\sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J (2m+2-K)} \simeq \frac{1}{2 C_{K-1}^{\frac{K}{2}-1}} \quad (157)$$

so that Eq 155 reads at leading order

$$K \text{ even} : G^R \simeq G e^{-\beta J K} 2 K C_{K-1}^{\frac{K}{2}-1} = G e^{-\beta J K} 2 \frac{K!}{(\frac{K}{2})! (\frac{K}{2}-1)!} \quad (158)$$

For instance for $K = 2$, one obtains

$$K = 2 : G^R \simeq 4 e^{-2\beta J} G \quad (159)$$

2. Leading behavior near zero temperature for odd K

When the branching ratio K is odd, then $(\frac{K}{2} - 1)$ is not an integer so that m cannot take this value, and the sum in the denominator is dominated by the contributions of the two closest integers $m = \frac{K-3}{2}$ and $m = \frac{K-1}{2}$

$$\begin{aligned} \sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J (2m+2-K)} &\simeq \sum_{m=0}^{K-1} \frac{e^{-\beta J |2m+2-K|}}{C_{K-1}^m} \simeq \frac{e^{-\beta J}}{C_{K-1}^{\frac{K-3}{2}}} + \frac{e^{-\beta J}}{C_{K-1}^{\frac{K-1}{2}}} \\ &= e^{-\beta J} \left[\frac{(\frac{K+1}{2})! (\frac{K-3}{2})!}{(K-1)!} + \frac{(\frac{K-1}{2})! (\frac{K-1}{2})!}{(K-1)!} \right] \\ &= e^{-\beta J} \frac{K (\frac{K-1}{2})! (\frac{K-3}{2})!}{(K-1)!} \end{aligned} \quad (160)$$

so that Eq 155 reads at leading order

$$K \text{ odd} : G^R \simeq G e^{-\beta J (K-1)} \frac{(K-1)!}{(\frac{K-1}{2})! (\frac{K-3}{2})!} = G e^{-\beta J (K-1)} (K-1) C_{K-2}^{\frac{K-1}{2}} \quad (161)$$

For instance for $K = 3$, one obtains

$$K = 3 : G^R \simeq 2 e^{-2\beta J} G \quad (162)$$

D. Conclusion for the equilibrium time $t_{eq}^{simple}(N)$ of a Cayley tree with N generations

Let us now consider a finite Cayley tree of branching ratio K with N generations. For the first RG step where $G_0 = 1$, we cannot use the perturbative analysis presented above to obtain G_1 . However since $G_1 \ll 1$ at low temperature, we may use the perturbative analysis given above to obtain the recursion

$$G_{n+1} \simeq \frac{G_n}{\rho(K)} \simeq \frac{G_1}{[\rho(K)]^n} \quad (163)$$

for all RG steps corresponding $1 \leq n \leq N-1$, where the factor $\rho(K)$ has been evaluated at low temperature (Eqs 158 and 161)

$$\begin{aligned} \rho(K) &= \frac{e^{\beta JK}}{2K C_{K-1}^{\frac{K}{2}-1}} = e^{\beta JK} \frac{(\frac{K}{2})! (\frac{K}{2}-1)!}{2(K!)} && \text{for even } K \\ &= \frac{e^{\beta J(K-1)}}{(K-1) C_{K-2}^{\frac{K-1}{2}-1}} = e^{\beta J(K-1)} \frac{(\frac{K-1}{2})! (\frac{K-3}{2})!}{(K-1)!} && \text{for odd } K \end{aligned} \quad (164)$$

Finally, at the last RG step, we could take into account that the center has $(K+1)$ neighbors instead of K , and has no further ancestor $B_a = 0$. However, this anomalous last step is only a boundary multiplicative contribution, as is G_1 , and cannot change the dependence upon the number N of generations for large N coming from Eq 163

$$G_N^{final} \propto \frac{1}{[\rho(K)]^N} \quad (165)$$

In summary, we obtain that the equilibrium time $t_{eq}^{simple}(N) = 1/(2G_N^{final})$ (Eq 77) of a Cayley tree of branching ratio K grows exponentially with the number N of generations

$$t_{eq}^{simple}(N) \propto [\rho(K)]^N \quad (166)$$

where the growth factor $\rho(K)$ is given explicitly by Eq 164 for any K .

E. Equilibrium time $t_{eq}^{Glauber}(N)$ for the Glauber dynamics

The above results concerning the simple dynamics can be extended to the Glauber dynamics as follows. The Hamiltonian H_{K+1}^{simple} of Eq. 108 has to be replaced for the first step by

$$H_{K+1}^{Glauber} \equiv \frac{1}{2 \cosh(\beta (\sum_{i=1}^K J \sigma_i^z + B_a))} \left(e^{-\beta \sigma^z (\sum_{i=1}^K J \sigma_i^z + B_a)} - \sigma^x \right) + \frac{1}{2 \cosh(\beta J \sigma^z)} \sum_{i=1}^K \left(e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^x \right) \quad (167)$$

Since the K leaves have no external field and are just linked to σ^z , the amplitude $\frac{1}{2 \cosh(\beta J \sigma^z)}$ reduces to the number $\frac{1}{2 \cosh(\beta J)}$. The remaining non-trivial amplitude $\frac{1}{2 \cosh(\beta (\sum_{i=1}^K J \sigma_i^z + B_a))}$ will disappear when we apply the perturbation method within the subspace annihilating the corresponding operator $(e^{-\beta \sigma^z (\sum_{i=1}^K J \sigma_i^z + B_a)} - \sigma^x)$. Our conclusion is thus that the equilibrium time $t_{eq}^{Glauber}(N)$ for the Glauber dynamics will have exactly the same leading exponential behavior in N as the result of Eq. 166 derived for the simple dynamics

$$t_{eq}^{Glauber}(N) \propto [\rho(K)]^N \quad (168)$$

even if the prefactor can differ (see the discussion on the differences between the equilibrium times of the simple and Glauber dynamics in Appendix B).

F. Comparison with previous results on dynamical barriers

From Eq. 166 and Eq. 168, we obtain that the energetic barrier $B_K(N)$ defined as the coefficient of β in $\ln t_{eq}(N)$

$$\begin{aligned} B_K(N) &= \lim_{\beta \rightarrow +\infty} \frac{\ln t_{eq}^{simple}(N)}{\beta} = NJK + O(1) && \text{for even } K \\ &= NJ(K-1) + O(1) && \text{for odd } K \end{aligned} \quad (169)$$

grows linearly with the number N of generations (i.e. logarithmically with the number of sites $\mathcal{N}_N \propto K^N$) in agreement with previous works of physicists [27–29] and of mathematicians [30–33]. Besides this correct scaling with N , it appears that the slope $(K-1)J$ for odd K of Eq. 169 coincides with the slope obtained in [28], where a so-called ‘disjoint strategy’ is optimal, whereas the slope KJ for even K of Eq. 169 differs from the slope $J(K-1)$ obtained in [27, 28], where a so-called ‘non-disjoint strategy’ is optimal. We refer to Refs [27–31] for more explanations on the differences between disjoint/non-disjoint strategies. For the present work, it is clear that the renormalization procedure making coherent clusters of spins within sub-trees corresponds to the disjoint strategy.

Finally, besides the Arrhenius factor involving the energetic barrier of Eq 169, the present renormalization procedure predicts explicit combinatorial prefactors for the exponential growth factor $\rho(K)$ (Eq. 164) that have not been previously discussed in the literature, to the best of our knowledge.

VII. CONCLUSION

In this paper, we have introduced a real-space RG procedure valid near zero-temperature to evaluate the largest relaxation time of classical random ferromagnets. We have used the standard mapping between the master equations satisfying detailed balanced and quantum Hamiltonians having an exact zero-energy ground state. The largest relaxation time t_{eq} governing the convergence of the dynamics towards the Boltzmann equilibrium is determined by the lowest non-vanishing eigenvalue $E_1 = 1/t_{eq}$ of the quantum Hamiltonian H . We have thus defined appropriate real-space RG rules for the quantum Hamiltonian to evaluate E_1 for finite systems. We have described how the renormalization flow can be explicitly solved for the two following cases.

(i) For the one-dimensional random ferromagnetic chain with free boundary conditions, the largest relaxation time t_{eq} can be expressed in terms of the set of random couplings for various choices of the dynamical transition rates. The validity of these RG results in $d = 1$ have been checked by comparison with another approach in Appendix.

(ii) For the pure Ising model on a Cayley tree of branching ratio K (coordination $(K+1)$), we have computed the exponential growth of $t_{eq}(N)$ with the number N of generations.

In a companion paper [34], we explain how the renormalization flow can be also explicitly solved for the Dyson hierarchical Dyson Ising model. In the future, we hope to obtain numerical results for the RG flow in finite dimensions $d > 1$.

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Appendix A: Check of the validity of the RG procedure in $d = 1$

In this Appendix, we present another approach to check the results of the RG procedure obtained in section V for the random ferromagnetic chain

1. Ansatz for the first excited quantum state in terms of exit probabilities

a. Eigenequation for ψ_1

For the quantum Hamiltonian \mathcal{H} corresponding to the spin-flip dynamics of classical spin models with the energy of Eq. 20, the exact ground state of Eq. 16

$$\psi_0(C) = \frac{e^{-\frac{\beta}{2}U(C)}}{\sqrt{Z}} = \frac{e^{-\frac{\beta}{2}\sum_{i<j}J_{ij}S_iS_j}}{\sqrt{Z}} \quad (\text{A1})$$

is invariant under a global flip of all the spins

$$\psi_0(-C) = \psi_0(C) \quad (\text{A2})$$

On the contrary, the first excited state will be antisymmetric under a global flip of all the spins

$$\psi_1(-C) = -\psi_1(C) \quad (\text{A3})$$

but its modulus is expected to coincide nearly with $\psi_0(C)$ in the two valleys around the two classical ferromagnetic ground states. It is thus convenient to set

$$\psi_1(C) = \psi_0(C)A(C) \quad (\text{A4})$$

and to look for the antisymmetric amplitude $A(C)$ (antisymmetric under a global flip of all the spins)

$$A(-C) = A(C) \quad (\text{A5})$$

The eigenvalue equation for the quantum Hamiltonian of Eq. 9,10, 11

$$\mathcal{H}|\psi_1\rangle = E_1|\psi_1\rangle \quad (\text{A6})$$

becomes via the change of variables of Eq. A4

$$[W_{out}(C) - E_1] A(C) = \sum_{C'} W(C \rightarrow C') A(C') \quad (\text{A7})$$

For a large system where E_1 is small, we expect that E_1 can be neglected with respect to $W_{out}(C)$ for all configurations different from the two classical ground states, so that one obtains the approximate equation

$$A(C) \underset{E_1 \rightarrow 0}{\simeq} \sum_{C'} \frac{W(C \rightarrow C')}{W_{out}(C)} A(C') = \sum_{C'} \pi_C(C') A(C') \quad (\text{A8})$$

where

$$\pi_C(C') \equiv \frac{W(C \rightarrow C')}{W_{out}(C)} = \frac{W(C \rightarrow C')}{\sum_{C''} W(C \rightarrow C'')} \quad (\text{A9})$$

represents the probability that the first exit from configuration C leads to C' for the master equation of Eq. 2, with the normalization

$$\sum_{C'} \pi_C(C') = 1 \quad (\text{A10})$$

b. Relation with exit probabilities

Exit probabilities are known to satisfy backward master equation similar to Eq. A8 (see for instance the textbooks [5–7]). More precisely, in a ferromagnet, one may introduce the probability $Q_+(C)$ that the dynamics starting in configuration C reaches first the configuration C_+ (all spins plus) than the configuration C_- (all spins minus). The complementary probability $Q_-(C) = 1 - Q_+(C)$ represents the probability that the dynamics starting in configuration C reaches first the configuration C_- than the configuration C_+ . The escape probability satisfies the backward master equation

$$Q_+(C) = \sum_{C'} \pi_C(C') Q_+(C') \quad (\text{A11})$$

for all configurations C different from the two ground states, and the boundary conditions

$$\begin{aligned} Q_+(C_+) &= 1 \\ Q_+(C_-) &= 0 \end{aligned} \quad (\text{A12})$$

This suggests the following Ansatz for the antisymmetric $A(C)$ satisfying Eq. A8 up to a normalization factor \mathcal{N}

$$A^{ansatz}(C) = \mathcal{N}(2Q_+(C) - 1) = \mathcal{N}(1 - 2Q_-(C)) \quad (\text{A13})$$

using $Q_+(C) = Q_-(-C)$ one obtains $A(-C) = -A(C)$.

The only point where $Q_+(C)$ does not satisfy Eq. A11 are the two boundaries C_+ and C_- where Q_+ is given by the b.c. corresponding to

$$\begin{aligned} \frac{A^{ansatz}(C_+)}{\mathcal{N}} &= 2Q_+(C_+) - 1 = 1 \\ \frac{A^{ansatz}(C_-)}{\mathcal{N}} &= 2Q_+(C_-) - 1 = -1 \end{aligned} \quad (\text{A14})$$

Let us now estimate E_1 for the Ansatz of Eq. A13 corresponding to

$$\psi_1^{Ansatz}(C) = \psi_0(C)A^{Ansatz}(C) = \mathcal{N}\psi_0^{Ansatz}(C)(2Q_+(C) - 1) \quad (A15)$$

via

$$\begin{aligned} E_1 &= \frac{\langle \psi_1^{Ansatz} | H_Q | \psi_1^{Ansatz} \rangle}{\langle \psi_1^{Ansatz} | \psi_1^{Ansatz} \rangle} \\ &= \frac{\sum_C \psi_0^2(C) A^{Ansatz}(C) [W_{out}(C) A^{Ansatz}(C) - \sum_{C'} W(C \rightarrow C') A^{Ansatz}(C')]}{\sum_C \psi_0^2(C) (A^{Ansatz}(C))^2} \end{aligned} \quad (A16)$$

In the numerator, all configurations C different from C_+ and C_- give zero-contributions as a consequence of Eq. A11. So the only contributions in the numerator come from $C = C_+$ and from $C = C_-$ where we may use the boundary conditions of Eq. A14 to obtain

$$\begin{aligned} E_1 &= \frac{\psi_0^2(C_+) \left[W_{out}(C_+) - \sum_{C'} W(C_+ \rightarrow C') (2Q_+(C') - 1) \right] + \psi_0^2(C_-) \left[W_{out}(C_-) - \sum_{C'} W(C_- \rightarrow C') (1 - 2Q_+(C')) \right]}{\sum_C \psi_0^2(C) (2Q_+(C) - 1)^2} \\ &= 2 \frac{\psi_0^2(C_+) [\sum_{C'} W(C_+ \rightarrow C') Q_-(C')] + \psi_0^2(C_-) [\sum_{C'} W(C_- \rightarrow C') Q_+(C')]}{\sum_C \psi_0^2(C) (2Q_+(C) - 1)^2} \end{aligned} \quad (A17)$$

The numerator involves the probability to reach first C_- before returning to C_+ when one leaves C_+ , and the probability to reach first C_- before returning to C_+ when one leaves C_+ , which are the same by symmetry.

2. Application to the random ferromagnetic chain near zero temperature

We now focus on the random ferromagnetic chain of N spins of Eq 83 with free boundary conditions for the two boundary spins S_1 and S_N . Near zero temperature (Eq 1), we may neglect the configurations containing more than one domain-wall, and work within the space of the following $(2N)$ configurations

$$\begin{aligned} |k >_N^{sym} &= \frac{1}{\sqrt{2}} [|S_1 = \dots = S_k = -1; S_{k+1} = \dots = S_N = +1 \rangle + |S_1 = \dots = S_k = 1; S_{k+1} = \dots = S_N = -1 \rangle] \\ |k >_N^{asym} &= \frac{1}{\sqrt{2}} [|S_1 = \dots = S_k = -1; S_{k+1} = \dots = S_N = +1 \rangle - |S_1 = \dots = S_k = 1; S_{k+1} = \dots = S_N = -1 \rangle] \end{aligned} \quad (A18)$$

where $k = 0, 1, \dots, N-1$. In physical terms, $|0 >^{sym}$ and $|0 >^{asym}$ are the symmetric and antisymmetric combination of the two ferromagnetic ground states where all spin have the same signs, whereas $|ksym >$ and $|kasym >$ with $1 \leq k \leq N-1$ are the symmetric and antisymmetric combination of the states where there exists a single domain-wall between the sites $(k, k+1)$.

We consider the quantum Hamiltonian

$$\begin{aligned} \mathcal{H}_N &= \sum_{2 \leq k \leq N-1} G(J_{k-1,k} \sigma_{k-1}^z + J_{k,k+1} \sigma_{k+1}^z) \left[e^{-\beta \sigma_k^z (J_{k-1,k} \sigma_{k-1}^z + J_{k,k+1} \sigma_{k+1}^z)} - \sigma_k^x \right] \\ &\quad + G(J_{1,2}) \left[e^{-\beta \sigma_1^z J_{1,2} \sigma_2^z} - \sigma_1^x \right] + G(J_{N-1,N} \sigma_{N-1}^z) \left[e^{-\beta \sigma_N^z J_{N-1,N} \sigma_{N-1}^z} - \sigma_N^x \right] \end{aligned} \quad (A19)$$

where the amplitudes G_k are given by a single even function $G(x) = G(-x)$ of the local field (see Eq. 34).

a. Two first eigenvectors within the single domain-wall approximation

The ground state $|\psi_0 >$ of zero energy is exactly known from Eq. A1

$$|\psi_0 >_N = \frac{1}{Z_N} \sum_{S_1, \dots, S_N} e^{\frac{\beta}{2} \sum_{i=1}^{N-1} J_{i,i+1} S_i S_{i+1}} |S_1, \dots, S_N > \quad (A20)$$

Within the reduced space of configurations containing no more than one domain-wall (Eq A18), the ground state reduces to

$$|\psi_0 >_N \simeq |0 >_N^{sym} + \sum_{k=1}^{N-1} e^{-\beta J_{k,k+1}} |k >_N^{sym} \quad (\text{A21})$$

near zero temperature

To respect the antisymmetry of Eq. A3, the first excited state will be a linear combination of the antisymmetric states of Eq. A18

$$|\psi_1 >_N \simeq |0 >_N^{asym} + \sum_{k=1}^{N-1} e^{-\beta J_{k,k+1}} A_N(k) |k >_N^{asym} \quad (\text{A22})$$

with some amplitudes $A_N(k)$ that we wish to determine. The eigenvalue equation for this first excited state of the Hamiltonian of Eq. A19 reads

$$\begin{aligned} 0 &= (\mathcal{H}_N - E_1) |\psi_1 >_N \\ &= |0 >_N^{asym} [-E_1 + f_1^- e^{-\beta J_{1,2}} (1 - A_N(1)) + f_N^- e^{-\beta J_{N-1,N}} (1 + A_N(N-1))] \\ &+ |1 >_N^{asym} [-E_1 e^{-\beta J_{1,2}} A_N(1) + f_1^- (A_N(1) - 1) + f_2^- e^{-\beta J_{2,3}} (A_N(1) - A_N(2))] \\ &+ \sum_{k=2}^{N-2} |k >_N^{asym} [-E_1 e^{-\beta J_{k,k+1}} A_N(k) + f_k^- e^{-\beta J_{k-1,k}} (A_N(k) - A_N(k-1)) + f_{k+1}^- e^{-\beta J_{k+1,k+2}} (A_N(k) - A_N(k+1))] \\ &+ |N-1 >_N^{asym} [-E_1 e^{-\beta J_{N-1,N}} A_N(N-1) + f_{N-1}^- e^{-\beta J_{N-2,N-1}} (A_N(N-1) - A_N(N-2)) + f_N^- (A_N(N-1) + 1)] \end{aligned} \quad (\text{A23})$$

in terms of the numbers

$$f_k^- \equiv G(J_{k-1,k} - J_{k,k+1}) \quad (\text{A24})$$

b. Ansatz with exit probabilities

Instead of solving exactly the eigenvalue problem of an $N \times N$ matrix of Eq. A23, we have proposed in section A 1 the following approximation : in all coefficients involving $|k >_N^{asym}$ with $k = 1, \dots, N-1$, we may neglect the term containing E_1 with respect to the others to obtain the $(N-1)$ equations for $k = 1, \dots, N-1$

$$f_k^- e^{-\beta J_{k-1,k}} (A_N^{ansatz}(k) - A_N^{ansatz}(k-1)) + f_{k+1}^- e^{-\beta J_{k+1,k+2}} (A_N^{ansatz}(k) - A_N^{ansatz}(k+1)) = 0 \quad (\text{A25})$$

with the following boundary conditions

$$\begin{aligned} A_N^{ansatz}(0) &= 1 \\ A_N^{ansatz}(N) &= -1 \end{aligned} \quad (\text{A26})$$

The only remaining term in Eq A23 is then the first line involving $|0 >_N^{asym}$ that determines the value of the energy E_1 as

$$E_1^{ansatz}(N) = f_1^- e^{-\beta J_{1,2}} (A_N^{ansatz}(0) - A_N^{ansatz}(1)) + f_N^- e^{-\beta J_{N-1,N}} (A_N^{ansatz}(N-1) - A_N^{ansatz}(N)) \quad (\text{A27})$$

So we have replaced the eigenvalue problem of Eq. A23 by a simpler homogeneous recurrence equation (Eq A25) with the boundary equations of Eq. A26, that can be solved as follows.

c. Exact solution for exit probabilities in one dimension

It is convenient to set as in Eq. A13

$$A_N^{ansatz}(k) = 2Q_0(k) - 1 = 1 - 2Q_N(k) \quad (\text{A28})$$

where $Q_N(k)$ satisfies

$$Q_N(k) = p_+(k)Q_N(k+1) + p_-(k)Q_N(k-1) \quad (\text{A29})$$

with the respective probabilities

$$\begin{aligned} p_+(k) &\equiv \frac{f_{k+1}^- e^{-\beta J_{k+1,k+2}}}{f_{k+1}^- e^{-\beta J_{k+1,k+2}} + f_k^- e^{-\beta J_{k-1,k}}} \\ p_-(k) &\equiv \frac{f_k^- e^{-\beta J_{k-1,k}}}{f_{k+1}^- e^{-\beta J_{k+1,k+2}} + f_k^- e^{-\beta J_{k-1,k}}} = 1 - p_+(k) \end{aligned} \quad (\text{A30})$$

and the boundary conditions

$$\begin{aligned} Q_N(0) &= 0 \\ Q_N(N) &= 1 \end{aligned} \quad (\text{A31})$$

Then $Q_N(k)$ represents the probability to reach first the boundary $k = N$ rather than the boundary $k = 0$ for a random walker starting at k and moving with probabilities of Eq. A30. The well-known solution of this standard problem can be obtained by recurrence [25] using Kesten variables [26] and reads

$$Q_N(k) = \frac{R(1,k)}{R(1,N)} \quad (\text{A32})$$

with

$$\begin{aligned} R(1,0) &= 0 \\ R(1,1) &= 1 \\ R(1,k \geq 2) &= 1 + \sum_{m=1}^{k-1} \prod_{n=1}^m \frac{p_-(n)}{p_+(n)} \\ R(1,N) &= 1 + \sum_{m=1}^{N-1} \prod_{n=1}^m \frac{p_-(n)}{p_+(n)} = 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \dots + \frac{p_-(1)p_-(2)\dots p_-(N-1)}{p_+(1)p_+(2)\dots p_+(N-1)} \end{aligned} \quad (\text{A33})$$

The corresponding estimate of the energy of Eq. A27 reads using Eq. A28

$$\begin{aligned} E_1^{\text{ansatz}}(N) &= f_1^- e^{-\beta J_{1,2}} ((1 - 2Q_N(0)) - (1 - 2Q_N(1))) + f_N^- e^{-\beta J_{N-1,N}} (1 - 2Q_N(N-1) - (1 - 2Q_N(N))) \\ &= 2f_1^- e^{-\beta J_{1,2}} (Q_N(1) - Q_N(0)) + 2f_N^- e^{-\beta J_{N-1,N}} (Q_N(N) - Q_N(N-1)) \\ &= 2 \frac{f_1^- e^{-\beta J_{1,2}} R(1,1) + f_N^- e^{-\beta J_{N-1,N}} [R(1,N) - R(1,N-1)]}{R(1,N)} \\ &= 2 \frac{f_1^- e^{-\beta J_{1,2}} + f_N^- e^{-\beta J_{N-1,N}} \left[\frac{p_-(1)p_-(2)\dots p_-(N-1)}{p_+(1)p_+(2)\dots p_+(N-1)} \right]}{1 + \sum_{m=1}^{N-1} \prod_{n=1}^m \frac{p_-(n)}{p_+(n)}} \end{aligned} \quad (\text{A34})$$

in terms of the ratios (Eq A30)

$$\frac{p_-(k)}{p_+(k)} = \frac{f_k^- e^{-\beta J_{k-1,k}}}{f_{k+1}^- e^{-\beta J_{k+1,k+2}}} = \frac{f_k^-}{f_{k+1}^-} e^{\beta(J_{k+1,k+2} - J_{k-1,k})} \quad (\text{A35})$$

Taking into account that absent links correspond to vanishing coupling $J_{0,1} = 0 = J_{N,N+1}$, one obtains

$$\left[\frac{p_-(1)p_-(2)\dots p_-(N-1)}{p_+(1)p_+(2)\dots p_+(N-1)} \right] = \frac{f_1^-}{f_N^-} e^{-\beta J_{1,2} + \beta J_{N-1,N}} \quad (\text{A36})$$

and

$$\begin{aligned}
R(1, N) &= 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \dots + \frac{p_-(1)p_-(2)\dots p_-(N-1)}{p_+(1)p_+(2)\dots p_+(N-1)} \\
&= 1 + \frac{f_1^-}{f_2^-} e^{\beta J_{2,3}} + \frac{f_1^-}{f_3^-} e^{-\beta J_{1,2} + \beta J_{2,3} + \beta J_{3,4}} + \frac{f_1^-}{f_4^-} e^{-\beta J_{1,2} + \beta J_{3,4} + \beta J_{4,5}} + \frac{f_1^-}{f_5^-} e^{-\beta J_{1,2} + \beta J_{4,5} + \beta J_{5,6}} \\
&\quad + \dots + \frac{f_1^-}{f_{N-1}^-} e^{-\beta J_{1,2} + \beta J_{N-2,N-1} + \beta J_{N-1,N}} + \frac{f_1^-}{f_N^-} e^{-\beta J_{1,2} + \beta J_{N-1,N}} \\
&= f_1^- e^{-\beta J_{1,2}} \left[\frac{e^{\beta J_{1,2}}}{f_1^-} + \frac{e^{\beta J_{1,2} + \beta J_{2,3}}}{f_2^-} + \frac{e^{\beta J_{2,3} + \beta J_{3,4}}}{f_3^-} + \frac{e^{\beta J_{3,4} + \beta J_{4,5}}}{f_4^-} + \dots + \frac{e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}}}{f_{N-1}^-} + \frac{e^{\beta J_{N-1,N}}}{f_N^-} \right] \\
&= f_1^- e^{-\beta J_{1,2}} \sum_{k=1}^N \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{f_k^-} \tag{A37}
\end{aligned}$$

so that the energy of Eq. A34 reads

$$\begin{aligned}
E_1^{ansatz}(N) &= 2 \frac{f_1^- e^{-\beta J_{1,2}} + f_N^- e^{-\beta J_{N-1,N}} \left[\frac{f_1^-}{f_N^-} e^{-\beta J_{1,2} + \beta J_{N-1,N}} \right]}{R(1, N)} = \frac{4 f_1^- e^{-\beta J_{1,2}}}{R(1, N)} \\
&= \frac{4}{\sum_{k=1}^N \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{f_k^-}} \tag{A38}
\end{aligned}$$

i.e. the equilibrium time reads using Eq A24

$$t_{eq}(N) = \frac{1}{E_1^{ansatz}(N)} = \frac{1}{4} \sum_{k=1}^N \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{f_k^-} = \frac{1}{4} \sum_{k=1}^N \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{G(J_{k-1,k} - J_{k,k+1})} \tag{A39}$$

in agreement with the RG result of Eq. 105 derived in the text.

3. Exact renormalization rules in configuration space for escape probabilities $Q_{\pm}(C)$

As a final remark, let us mention the link with previous works concerning renormalization rules *in configuration space*. As explained in [35], backward master equations satisfy exact renormalization rules in configuration space. Upon the elimination of the configuration \mathcal{C}_0 , the surviving configurations \mathcal{C} satisfy the same equation as before (Eq A11)

$$W_{out}^R(C) Q_+(C) = \sum_{C'} W^R(C \rightarrow C') Q_+(C') \tag{A40}$$

where the renormalized transitions rates W^R evolve with the RG equations

$$\begin{aligned}
W^{Rnew}(\mathcal{C} \rightarrow \mathcal{C}') &= W^R(\mathcal{C} \rightarrow \mathcal{C}') + \frac{W^R(\mathcal{C} \rightarrow \mathcal{C}_0) W^R(\mathcal{C}_0 \rightarrow \mathcal{C}')}{W_{out}^R(\mathcal{C}_0)} \\
W_{out}^{Rnew}(\mathcal{C}) &= W_{out}^R(\mathcal{C}) - \frac{W^R(\mathcal{C} \rightarrow \mathcal{C}_0) W^R(\mathcal{C}_0 \rightarrow \mathcal{C})}{W_{out}^R(\mathcal{C}_0)} \tag{A41}
\end{aligned}$$

These RG rules for backward master equations are exact and can be used [35], but only for small sizes as a consequence of the exponential growth of configurations. The RG rules of Eq. A41 have been first derived via a Strong Disorder RG approach [36].

Note that the RG rules of Eq. A41 can be rewritten directly for the renormalized probabilities

$$\pi_C^R(C') \equiv \frac{W^R(C \rightarrow C')}{W_{out}^R(C)} \tag{A42}$$

that evolve according to

$$\pi_C^{Rnew}(C') \equiv \frac{W^{Rnew}(C \rightarrow C')}{W_{out}^{Rnew}(C)} = \frac{\pi_C^R(C') + \pi_C^R(C_0) \pi_{C_0}^R(C')}{1 - \pi_C^R(C_0) \pi_{C_0}^R(C)} \tag{A43}$$

Appendix B: Dependence on the choice of the dynamics

In this Appendix, we describe how the equilibrium time depends on the choice of the dynamics satisfying detailed balance

1. Case of the Glauber dynamics

For the Glauber dynamics, one expects that the dynamical barrier coincides with the maximal energy cost on the optimal path between the two ground states. For instance for the one-dimensional random ferromagnetic chain, the result of Eq. 106 satisfies

$$\frac{1}{\beta} \ln [t_{eq}^{Glauber}(N)] = \max_{0 \leq k \leq N-1} (2J_{k,k+1}) = \max_{0 \leq k \leq N} \left(U_N^{(k,N-k)} - U_N^{GS} \right) \quad (B1)$$

where $U_N^{(k,N-k)}$ represents the energy of the configuration where the first k spins are (-1) , whereas all others spins are $(+1)$.

To better understand the differences with the simple dynamics described below, it is useful to write the result of Eq. 106 for the two smallest sizes, with $N = 2$ and $N = 3$ spins

$$\begin{aligned} t_{eq}^{glauber}(N=2) &= \frac{1}{2} [1 + e^{2\beta J_{1,2}}] \\ t_{eq}^{glauber}(N=3) &= \frac{1}{2} [1 + e^{2\beta J_{1,2}} + e^{2\beta J_{2,3}}] \end{aligned} \quad (B2)$$

2. Case of the simple dynamics

For the 'simple' dynamics, the correspondence of Eq. B1 between the dynamical barrier and the maximal energy cost of a single domain wall does not hold, as can be seen already for the one-dimensional case with $N = 2$ and $N = 3$ spins since Eq 94 reads

$$\begin{aligned} t_{eq}^{simple}(N=2) &= \frac{1}{2} e^{\beta J_{1,2}} \\ t_{eq}^{simple}(N=3) &= \frac{1}{4} [e^{\beta J_{1,2}} + e^{\beta(J_{1,2}+J_{2,3})} + e^{\beta J_{2,3}}] \end{aligned} \quad (B3)$$

For $N = 2$, the difference by a factor of 2 between the dynamical barriers can be understood from the differences between the transitions rates for the simple dynamics (Eq 24)

$$\begin{aligned} W^{simple}(++ \rightarrow +-) &= W^{simple}(++ \rightarrow -+) = e^{-\beta J_{1,2}} \\ W^{simple}(+- \rightarrow ++) &= W^{simple}(+- \rightarrow --) = e^{+\beta J_{1,2}} \end{aligned} \quad (B4)$$

and for the Glauber dynamics (eq 26)

$$\begin{aligned} W^{Glauber}(++ \rightarrow +-) &= W^{Glauber}(++ \rightarrow -+) = \frac{e^{-\beta J_{1,2}}}{e^{+\beta J_{1,2}} + e^{-\beta J_{1,2}}} = \frac{e^{-2\beta J_{1,2}}}{1 + e^{-2\beta J_{1,2}}} \\ W^{Glauber}(+- \rightarrow ++) &= W^{Glauber}(+- \rightarrow --) = \frac{e^{+\beta J_{1,2}}}{e^{+\beta J_{1,2}} + e^{-\beta J_{1,2}}} = \frac{1}{1 + e^{-2\beta J_{1,2}}} \end{aligned} \quad (B5)$$

For $N = 2$ spins, the equilibrium time is determined by the rate $W(++ \rightarrow +-)$ to create a domain-wall when starting from one ground state (the time to eliminate the domain-wall is then negligible), and these two rates are respectively of order $e^{-\beta J_{1,2}}$ for the simple dynamics and of order $e^{-2\beta J_{1,2}}$ for the Glauber dynamics. One could argue that the Glauber dynamics is more 'physical', in the sense that all transitions rates remain bounded near zero-temperature, whereas in the 'simple' dynamics transition rates corresponding to a decrease of the energy diverge near zero temperature. Nevertheless, one expects on physical grounds that the difference between the dynamical barriers of the two dynamics remains of order $O(1)$, as found in this article for the one-dimensional case and for the tree case, and as found in the companion paper [34] for the Dyson hierarchical Ising model.

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